

Phenomenology of the Term Structure of Interest Rates with Padé Approximants ¹

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Abstract

The classical approach in finance attempts to model the term structure of interest rates using specified stochastic processes and the no arbitrage argument. Up to now, no universally accepted theory has been obtained for the description of experimental data. We have chosen a more phenomenological approach. It is based on results obtained some twenty years ago by physicists, results which show that Padé Approximants are very suitable for approximating large classes of functions in a very precise and coherent way. In this paper, we have chosen to compare Padé Approximants with very low indices with the experimental densities of interest rates variations. We have shown that the data published by the Federal Reserve System in the United States are very well reproduced with two parameters only. These parameters are rather simple functions of the lag and of the maturity and are directly related to the moments of the distributions.

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1 Introduction

The classical approach in finance attempts to model the term structure of interest rates using specified stochastic processes and the no arbitrage argument. Numerous theoretical models (see for example [31], [6], [15], [16]) have been proposed. Although they provide analytical formulas for the pricing of interest rate derivatives, the implied deformations of the term structure have a Brownian motion component and are often rejected by empirical data [5].

The inadequacies of the Gaussian model for the description of financial time series has been reported for a long time [19], [20] but the availability of enormous sets of financial data, with a small time scale, has renewed the interest in the subject (see [27], [23], [13], [25]). In particular, the fat-tail property of the empirical distributions of price changes has been widely documented and is a crucial feature for monitoring the extreme risks [10], [17]. Most of the recent studies concern stock indexes or exchange rates [25], [13], [27], with high frequency data. Studies on interest rates are rather rare and often limited to a few maturities (three and six months cash rates in [28], Bund futures in [2]). The paper [3] is an exception as the US forward rate curve with maturities up to four years is modeled.

There is a long history of attempts to model price fluctuations with Levy distributions [19]. The family of Levy laws exhibits qualitatively heavy tails and enjoys the suitable properties of stable laws [2], [12]. However, they are defined in an unduly complicated manner (using the characteristic function) and do not even have a finite moment of order two (variance). Truncated Levy distributions have been proposed to circumvent this drawback [23], [21]. Other alternative distributions have also been proposed (see [18], [9], [26]).

Our aim, in this paper, is to propose an alternative description of the empirical distributions of the variations of interest rates in a simple form, with a small number of parameters but enjoying the property of providing a remarkable fit of the full distribution and hence, at the same time, of both the central part and the tails.

Twenty five years ago, physicists have started studying sets of reduced rational forms which are particularly well suited to approximate, in a very uniform way, functions which belong to rather large classes. These forms are called Padé Approximants. Many major progresses in the understanding of these approximants have been made at that time. Since there is a very extensive literature concerning the validity and the usefulness of these approximants, which are intrinsically related to the theory of rational polynomials, to the theory of continuous fractions and to the theory of moments, we refer the reader directly to some relevant books [14], [1], [4] where more references can be found. We restrict ourselves, here, to a short presentation of the ideas leaving out the fine points. The basic idea of the Padé Approximants is to approximate a large class of functions of, say, the variable v by the ratio of two polynomials $T^M(v)$ and $B^N(v)$ of respective degrees

M and N .

In this paper, we propose to apply the Padé Approximants to the study of daily interest rate changes and show their relevance on data obtained from the American bond market from 1977 to 1997 [11].

In Section 2, we discuss the discreteness of the data and their approximation by continuous distributions. In Section 3, some general properties of the Padé Approximants are outlined. Section 4 contains a justification of the criteria chosen in order to assess the quality of the fit. The data are analyzed and our results are presented in Section 5. Section 6 contains the conclusions and some final remarks.

2 Interest Rates

Consider the term structures of interest rates given by samples of N_{tot} daily interest rates for constant maturities $[m]$. Let us call

$$I^{[m]}(t), \quad t = 1, \dots, N_{tot}, \quad (1)$$

these spot interest rates where the upper index $[m]$ (in unit year) specifies the maturity and t indexes, chronologically, the opening days.

A short digression on our notation for the “units” of interest rates is useful at this point. Interest rates are canonically, like in the data base we used, given in “percent/year” (which we will henceforth denote $[\%/year]$ and which is usually and abusively simply called “percent”) with two meaningful digits after the decimal point. Hence, the natural unit we will use for interest rate variations is the basis point $10^{-4}/year$ which we denote $[b.p./year]$. The inverse unit is $[year/b.p.]$ will also be used. The daily changes in interest rates are obviously also recorded as integers in units $[b.p./year]$ and vary from roughly -100 to $+100$. for a maturity of one year (this interval may be larger for higher lags). The unit $[b.p./year]$ is the natural unit to perform discrete summations and to expose the problem most simply. However, at the end, the parameters entering the problem and determined from the data are recorded in the $[\%/year]$ (and its inverse unit) better adapted to their actual values.

See Appendix A for a detailed discussion of the dimensions of the variables and of the parameters we will use and hence of their behaviour when units are changed.

2.1 Variations of the Interest Rates

From the interest rates $I^{[m]}(t)$ (integers in units $[b.p./year]$), variations of the interest rates $\delta I_L^{[m]}(t)$ at a lag of L days can be defined for each maturity $[m]$ as

$$\delta I_L^{[m]}(t) = I_L^{[m]}(t + L) - I_L^{[m]}(t), \quad t = 1, \dots, (N_{tot} - L). \quad (2)$$

These (overlapping) variations of interest rates are expressed again as integers in units [b.p./year]. Our main goal is to study and to compare the distributions of $\delta I_L^{[m]}(t)$ for all available maturities and for all time lags which we have chosen in the range from $L = 1$ to $L = 30$ days.

2.2 Distribution of the Variations of the Interest Rates

For every variation equal to a given value \hat{v} in units [b.p./year] (where \hat{v} is an integer), we can count the number of times this variation occurs in the experimental sample. We denote these distributions of variations by $N_L^{[m]}(\hat{v})$

$$\begin{aligned} N_L^{[m]}(\hat{v}) &\equiv \text{number of times } \delta I_L^{[m]}(t) = \hat{v} \\ &= \text{card } \{t, \delta I_L^{[m]}(t) = \hat{v}\} . \end{aligned} \quad (3)$$

By construction, we have $N_L^{[m]}(\hat{v})$ equal to zero outside the range $[\hat{v}_{min}, \hat{v}_{max}]$

$$\begin{aligned} \hat{v}_{min} &= \min_t \{ \delta I_L^{[m]}(t) \} \\ \hat{v}_{max} &= \max_t \{ \delta I_L^{[m]}(t) \} \end{aligned} \quad (4)$$

and

$$\sum_{\hat{v}=\hat{v}_{min}}^{\hat{v}_{max}} N_L^{[m]}(\hat{v}) = N_{tot} - L . \quad (5)$$

Obviously, \hat{v}_{min} and \hat{v}_{max} depend on the maturity $[m]$ and on the lag L but, to simplify the formulas, we have suppressed these extra labels.

At this point, it is worth making two comments.

1. The opening days are not always consecutive, due to the presence of Sundays and other non working days. We have considered that the definition of the lag ignores these gaps. We work with a business time scale rather than a physical one. A more careful study could take into account this fact and limit itself to consider as belonging to lag one only those days which are non separated by one or more missing days. In order to keep our useful data as large as possible we have chosen to ignore this difficulty. A more refined discussion could obviously take this into account at the price of a lowering in statistics. With high frequency data, the definition of a time scale that models the market activity is even more crucial [7].
2. For lags greater than one, the distributions we have defined take into account overlapping periods. For example for lag two, one could have defined two distributions by using non overlapping two days periods: the distribution of the even days and the distribution of the odd days. The number of points in these two distributions is divided by two (for lag L , it would

be divided by L) decreasing the statistics (in a too drastic way when L is larger than four or five days for our limited sample). This would however enable one to study autocorrelations and volatility propagation. In [27], the authors made a precise study of these overlapping effects, varying the overlapping intervals. Their results were never systematically affected by these variations. With overlapping periods, the increments are clearly not independent. However, we will assume that they are identically distributed at a given time scale L . This unconditional approach is motivated by the fact that extreme events occur infrequently and do not seem to exhibit time dependence [8].

The empirical discretized density function $\hat{f}_L^{[m]}(\hat{v})$ of the variations of the interest rates \hat{v} for every maturity $[m]$ and every lag L is defined by

$$\hat{f}_L^{[m]}(\hat{v}) = \frac{N_L^{[m]}(\hat{v})}{N_{tot} - L}. \quad (6)$$

Again, $\hat{f}_L^{[m]}(\hat{v})$ is defined for integer values of \hat{v} , zero outside the relevant $[\hat{v}_{min}, \hat{v}_{max}]$ range and normalized

$$\sum_{\hat{v}=\hat{v}_{min}}^{\hat{v}_{max}} \hat{f}_L^{[m]}(\hat{v}) = 1. \quad (7)$$

The analogous continuous densities $f(v)$ are continuous functions of the continuous variations v . We chose [b.p./year] again as the unit of v . The corresponding unit for f is the inverse [year/b.p.]. They are normalized in such a way that their integral be equal to one. For the experimental discretized distributions (6), the normalizations are equivalent to stepwise integrals

$$\text{Normalization}_{\text{data}} \equiv \int_{v_{min}}^{v_{max}} f_L^{[m]}(v) dv = \sum_{\hat{v}=\hat{v}_{min}}^{\hat{v}_{max}} \left(\hat{f}_L^{[m]}(\hat{v}) \right) = 1. \quad (8)$$

Obviously, for continuous variations v , the density $f_L^{[m]}(v)$ takes the constant value $\hat{f}_L^{[m]}(\hat{v})$ in the interval of length 1 centered around the discretized variation \hat{v} . Precisely

$$f_L^{[m]}(v) = \hat{f}_L^{[m]}(\hat{v}) \quad (9)$$

for

$$\left(\hat{v} - \frac{1}{2} \right) < v \leq \left(\hat{v} + \frac{1}{2} \right). \quad (10)$$

Outside the $[v_{min}, v_{max}]$ range is f taken to be zero so that the limit in the integral can be taken to be $-\infty$ and $+\infty$.

The behaviour of these quantities under changes in the units is outlined in Appendix A.

The moments of order two of the distributions are again defined by a step-wise integration. To a very good approximation, since the mean values of the distributions are essentially zero, these second moments can be identified with the variance

$$\text{Variance}_{\text{data}} \equiv \int_{-\infty}^{\infty} v^2 f_L^{[m]}(v) dv = \sum_{\hat{v}=\hat{v}_{min}}^{\hat{v}_{max}} \left(\hat{v}^2 \hat{f}_L^{[m]}(\hat{v}) \right). \quad (11)$$

These variances (obtained in units [b.p./year]²) are obviously finite by construction. We however hope that the true variance, which is in principle less securely computed from the data than the normalization (8), is correctly estimated by the discretized computation (11). In other words, we conjecture that the true distribution is of finite variance contrary to some theoretical distributions like the Levy distributions which have been proposed in the literature (see [22], [24], [2]) and can lead to an infinite variance.

Later in the paper, we will discuss the variance and will also comment on the higher moments (skewness, kurtosis) of the distributions.

2.3 General Comments

We expect that the distributions of the variation of interest rates in the continuous variable are smooth functions of v and hence should be approximated by (normalized) continuous functions. These distributions are essentially maximum for a variation $v = 0$ and decrease on both sides i.e. in the directions $v \rightarrow +\infty$ and $v \rightarrow -\infty$. We also expect that the distributions are, in first approximation, symmetrical around $v = 0$ and hence even functions. These conjectures turn out to be very well substantiated by the experimental facts. As we will see however, we have decided to allow for a very slight shift s in the axis of symmetry. The distribution will then enjoy the symmetry $(v - s) \leftrightarrow -(v - s)$. The extra parameter s will turn out to be practically zero (less than a few [b.p./year] and rarely different from zero at conventional confidence level, see Table 2 and Table 3). In the next section we will give, in a more precise way, our notation and our basic hypothesis.

3 Padé Approximants

In this section, we fix the notation which will be used for the Padé parameters. We also discuss, for the approximants we are going to use, their normalization, their variance and their positivity.

3.1 Presentation of the Padé Approximants

Padé Approximants $P^{[M,N]}(v)$ are rational functions of a continuous variable v of the form

$$P^{[M,N]}(v) = \frac{T^M(v)}{B^N(v)} \quad (12)$$

and are indexed by the degrees M and N of the polynomials T^M and B^N appearing in the numerator (Top) and the denominator (Bottom) of the rational fraction P .

These rational functions are used to obtain good approximations to the continuous distributions (8) defined in the introduction. Since the distributions are real, we choose the polynomials T and B to depend on real parameters only.

As a general requirement, we have to limit ourselves, at the end, to values of the parameters of the Padé Approximants such that the densities are purely positive. Indeed, even extremely small negative probabilities in the tails would not make much sense and should be excluded.

As we have insisted on in the introduction, we expect that the Padé Approximants (and hence the two polynomials T and B) are symmetrical under a left-right symmetry around an axis shifted at most slightly (by s) from the $v = 0$ axis. More precisely, we will postulate that the two polynomials are even functions of the variable $(v - s)$ i.e. functions of $(v - s)^2$ and hence that M and N are even numbers.

We have also commented about the fact that the Padé Approximants should be normalized

$$\text{Normalization}_{\text{Padé}} \equiv \int_{-\infty}^{+\infty} P^{[M,N]}(v) dv = 1 \quad (13)$$

and have a finite variance

$$\text{Variance}_{\text{Padé}} \equiv \int_{-\infty}^{+\infty} P^{[M,N]}(v)(v - s)^2 dv < \infty. \quad (14)$$

The expectation (mean value) of the Padé distribution is simply

$$\text{Mean Value}_{\text{Padé}} \equiv \int_{-\infty}^{+\infty} P^{[M,N]}(v)v dv = s. \quad (15)$$

This mean value will turn out to be extremely small in absolute value for our experimental samples.

The two general requirements (13) and (14) imply that the even degrees M and N in (12) cannot be chosen arbitrarily. We need

$$M + 4 \leq N. \quad (16)$$

On the other hand, though our experimental sample is reasonably large we do not wish to introduce too many parameters in the fit. Indeed

- It is known that for many classes of functions even low values of M and N will give very good fits
- We also expect that the resulting distribution should be rather smooth. Numerical minimizations with too many parameters will push the routines to mimic fake oscillations which are the result of the fact that the sample is of finite size. These oscillations do not correspond to any intrinsic structure.

Hence we have chosen to limit ourselves to $N = 8$ and hence from (16) to $M = 4$. The relevant Padé Appriximants belong to the categories $[0, 4]$, $[0, 6]$, $[0, 8]$, $[2, 6]$, $[2, 8]$, $[4, 8]$. They can all be written at once in the form $[4, 8]$. Restricting oneself to some parameters equated to zero will produce the other forms. If we allow for infinite variance, the Padé $[0, 2]$, $[2, 4]$ and $[4, 6]$ could also be considered.

We will now write the general case in a completely explicit form. This allows us not only to fix the notation but also to profit from well-known integration formulas for ratios of polynomials [30].

To parametrize the numerator of the Padé function, it is convenient to introduce a complex polynomial $U^2(v)$ of second degree

$$U^2(v) = u_0 \left(1 + iu_1(v - s) + u_2(v - s)^2 \right) \quad (17)$$

and write the numerator of $P^{[4,8]}$ as

$$\begin{aligned} T^4(v) &= (U^2(v))^* U^2(v) \\ &= u_0^2 (1 + (u_1^2 + 2u_2)(v - s)^2 + u_2^2(v - s)^4) \\ &\equiv n_0 + n_2(v - s)^2 + n_4(v - s)^4 \end{aligned} \quad (18)$$

which depend on three real parameters $u_i, i = 0, 1, 2$. When M is decreased to 2, $u_2 = 0$ and when M is decreased to 0, both $u_1 = u_2 = 0$.

For reasons which will be explained shortly, it is even more convenient to split the denominator $B^8(v)$ of the general Padé $P^{[4,8]}$ (12) in the form of the modulus square of the complex polynomials $Q^4(v)$ of degree 4.

$$Q^4(v) = 1 + iq_1(v - s) + q_2(v - s)^2 + iq_3(v - s)^3 + q_4(v - s)^4 \quad (19)$$

where the four parameters $q_i, i = 1, \dots, 4$ are real. We then obtain the purely real eighth's degree denominator

$$\begin{aligned} B^8(v) &= (Q^4(v))^* Q^4(v) \\ &= 1 + (q_1^2 + 2q_2)(v - s)^2 + (q_2^2 + 2q_4 - 2q_1q_3)(v - s)^4 \\ &\quad + (q_3^2 + 2q_2q_4)(v - s)^6 + q_4^2(v - s)^8 \\ &\equiv 1 + d_2(v - s)^2 + d_4(v - s)^4 + d_6(v - s)^6 + d_8(v - s)^8. \end{aligned} \quad (20)$$

It is clear that the term of zero degree in this denominator has been chosen to be equal to 1 without any loss of generality. When N is decreased to 6, $q_4 = 0$ and when N is decreased to 4, $q_4 = q_3 = 0$.

Since the form of the Padé we have chosen, which is an even real function of $(v - s)$ only, is the modulus square of a complex quantity, the positivity of the distribution is guaranteed whatever be the parameters chosen in U^2 and Q^4 .

3.2 Normalization, Variance and Positivity

The form (20) may be thought, at first sight, to be unduly complicated. In fact, it has a very important advantage. Indeed, the normalization and the variance can be computed analytically [30]. First, the normalization (13) is

$$\text{Normalization}_{\text{Padé}[4,8]} = \pi \frac{(q_2 q_3 - q_1 q_4) n_0 - q_3 n_2 + q_1 n_4}{q_1 q_2 q_3 - q_1^2 q_4 - q_3^2}. \quad (21)$$

Imposing that this normalization be 1 decreases the number of arbitrary parameters by one.

From the expression (21) valid for the $P^{[4,8]}$, the normalization of the Padé with lower M and N can easily be deduced by suitable limiting procedures. Since we will mainly use $P^{[0,4]}$ and $P^{[0,6]}$, let us quote explicitly these particular cases

$$\text{Normalization}_{\text{Padé}[0,6]} = -\pi \frac{q_2 n_0}{q_3 - q_1 q_2}. \quad (22)$$

and

$$\text{Normalization}_{\text{Padé}[0,4]} = \pi \frac{n_0}{q_1} \quad (23)$$

which for a normalized $P^{[0,4]}$ determines n_0 as

$$n_0 = \frac{q_1}{\pi}. \quad (24)$$

The variance (14) is

$$\text{Variance}_{\text{Padé}[4,8]} = \pi \frac{-q_3 q_4 n_0 + q_1 q_4 n_2 + (q_3 - q_1 q_2) n_4}{q_4 (q_1 q_2 q_3 - q_1^2 q_4 - q_3^2)}. \quad (25)$$

For $P^{[0,6]}$ this reduces to

$$\text{Variance}_{\text{Padé}[0,6]} = \pi \frac{n_0}{q_3 - q_1 q_2} \quad (26)$$

and for $P^{[0,4]}$ to

$$\begin{aligned} \text{Variance}_{\text{Padé}[0,4]} &= -\pi \frac{n_0}{q_1 q_2} \\ &= -\frac{1}{q_2} \end{aligned} \quad (27)$$

where the last line in the equation refers to a normalized Padé $[0, 4]$ (24).

To be complete, let us also give the expression for the kurtosis. Obviously, in order to have a finite kurtosis, the Padé have to be restricted to $P^{[0,6]}$, $P^{[0,8]}$ and $P^{[2,8]}$.

$$\text{Kurtosis}_{\text{Padé}[2,8]} = \pi \frac{q_1 q_4 n_0 + (q_3 - q_1 q_2) n_2}{q_4 (q_1 q_2 q_3 - q_1^2 q_4 - q_3^2) \times \text{Variance}^2}. \quad (28)$$

The excess kurtosis is given by subtracting 3 to the kurtosis. The kurtosis for $P^{[0,6]}$ reduces to

$$\begin{aligned} \text{Kurtosis}_{\text{Padé}[0,6]} &= -\frac{1}{\pi} \frac{q_1 (q_3 - q_1^2)}{q_3 n_0} \\ &= \frac{q_1 q_2}{q_3}. \end{aligned} \quad (29)$$

The last line is for a normalized $P^{[0,6]}$.

Let us end this section by some technical points

- First, let us state again that the positivity of the distribution follows trivially from our writing of the Padé function in terms of the modulus square of the ratio U^2/Q^4 .
- Let us note that, in order to obtain the values of the normalization, the variance and the kurtosis starting from the formula given in the literature (see for example [30]), an analytical continuation has been performed passing from real to complex parameters in Q . A short discussion is given in appendix C. We have checked, in each instance, that the conditions allowing the analytical forms are satisfied.
- Technically, Padé Approximants, though they often approximate data very well and enjoy numerous properties are not “robust”, in the following sense. When the powers M and N are changed, the optimal values of the Padé parameters approximating in the best way the data (see (30) below) can vary widely. In other words, the optimal value of the parameters for a given choice of M and N do not predict, in general, in a very simple way, the values of the parameters for other choices of M and N . We refer the reader to the relevant literature about this point, which has been studied widely [14], [4].

However, for a given value of M and N , the parameters entering the Padé vary smoothly with respect to the lag and to the maturity. Hence the values for different lags and maturities can be compared and fitted extremely well to simple forms.

As we will show, the normalized experimental distributions $f(v)$ ((6), (9)) will be extremely well approximated by the normalized $P^{[0,4]}$ Padé Approximants.

4 Criteria, χ^2

In order to estimate the parameters appearing in the Padé Approximants, a measure of the distance between the experimental distribution and the Padé distribution should be chosen and minimized.

We have decided to perform a least square fit by defining for the (non-normalized) distributions N (see (3),(5)) a χ^2

$$\chi^2 = \sum_{\hat{v}=\hat{v}_{min}-R}^{\hat{v}_{max}+R} \frac{(P_{bare}(\hat{v}) - N(\hat{v}))^2}{\sigma(\hat{v})^2} \quad (30)$$

which is discussed in the items below.

Minimization of the χ^2 will be used to determine the estimates of the parameters n_i and q_i entering in the Padé Approximants. In the formula (30), we have, for simplicity, suppressed all the indices referring to the lag L , the maturity $[m]$ and the Padé indices $[M, N]$.

Let us explain and comment on this formula as applied to our case

- $P_{bare}(\hat{v})$ is the relevant Padé Approximant computed at the relevant position but normalized to the total number of data points instead of one. The criteria itself is a pure number and hence independent of the unit chosen to make the computations. If the bare $N(\hat{v})$ are arbitrarily multiplied by μ both the bare P 's and the σ 's should be multiplied by the same μ .

The normalized Padé Approximant $P(\hat{v})$ is thus given in terms of the bare $P_{bare}(\hat{v})$ as

$$P(\hat{v}) = \frac{P_{bare}(\hat{v})}{N_{tot} - L} \quad (31)$$

in units [year/b.p.] for the variable \hat{v} expressed in units [b.p./year].

- The sum runs on the integer discrete integer positions \hat{v} (in units [b.p./year] which we have chosen to be our natural binning) extending at least from the first instance \hat{v}_{min} where $N(\hat{v})$ is non zero to the last position \hat{v}_{max} where it is non zero. The inclusion of the integer $R \geq 0$ in (30) allows one to take into account in the summation an extended range of \hat{v} where we know experimentally that $N(\hat{v})$ is zero. Evidently, the summation in (30) could or should in principle be extended to the full domain $[-\infty, +\infty]$. Outside the initial range $[\hat{v}_{min}, \hat{v}_{max}]$, $N(\hat{v})$ is zero and $P(\hat{v})$ is small but non zero. In fact, restricting oneself to the range $[\hat{v}_{min} - R, \hat{v}_{max} + R]$ does not change the fits very significantly. For large absolute values of \hat{v} , many bins have zero observations. For this reason, the empirical characterisation of the probability laws is sometimes performed with cumulative distributions (see [2]). In order to keep the implementation as simple as possible and to avoid numerical integrations, we have decided to deal with probability densities.

- In order to make sense we have also to chose the “errors” $\sigma(\hat{v})$ in the data in a suitable way. Normally, in the absence of knowledge on the experimental errors on the $N(\hat{v})$, it is customary to chose all the $\sigma(\hat{v})$ equal to the same value. Indeed, the “experiment” is a one time experiment and cannot be reproduced at will.

However, in our case, we think that we can argue that we have some indications on the errors. If the interest rates were purely statistically produced, we would expect, in first approximation a random distribution around the theoretical curve and $\sigma(\hat{v})$ for given \hat{v} to be of the order of $\sqrt{N(\hat{v})}$, i.e.

$$\text{if } N(\hat{v}) \neq 0 \text{ then } \sigma(\hat{v}) = \sqrt{N(\hat{v})}. \quad (32)$$

This is not exactly the case but seems to be a reasonable first guess. Small variations about this value, due to the non randomness of the data are obviously not important and will influence very little the determination of the parameters obtained by minimizing the χ^2 . When $N(\hat{v}) = 0$, the appearance of $\sqrt{N(\hat{v})} = 0$ in the denominator does not make sense. We then naturally chose $\sigma(\hat{v}) = 1$, for the \hat{v} such that $N(\hat{v}) = 0$ (the minimal error on an integer number)

$$\text{if } N(\hat{v}) = 0 \text{ then } \sigma(\hat{v}) = 1. \quad (33)$$

In the following section, we apply the ideas of the two last sections to our data set.

5 Data Analysis

5.1 Presentation and Statistical Description of the Data

The raw data we have used are the American daily spot interest rates for constant maturities equal to one, two, three, five, seven, ten and thirty years

$$[m] = [1], [2], [3], [5], [7], [10], [30] \quad (34)$$

between February 15, 1977 and August 4, 1997. Altogether 7 rates for each of the

$$N_{tot} = 5108 \quad (35)$$

opening days considered. They are calculated from bond prices, published by the Board of Governors of the Federal Reserve System [11] in the United States of America and freely available on their Web site.

In Table 1, we have given some basic statistical results which can be obtained directly from the data : in particular the mean , the variance, the skewness and

the kurtosis for all available maturities and for lags $L = 1, 5, 10, 15, 20, 25, 30$. We have restricted ourselves to rather small time scales (up to thirty business days). For longer periods, the sizes of the samples become too small and the mean values cannot be neglected safely. All the variations in interest rates exhibit the same general behaviour : increasing variance and decreasing leptokurticity for larger lags. The small assymetries in the data are obviously not reproduced by the even Padé Approximants that we have selected.

5.2 Generalities

Taking into account our general consideration given above, the precise χ^2 we have chosen to minimize is

$$\chi^2 = \sum_{\hat{v}=\hat{v}_{min}-R}^{\hat{v}_{max}+R} \frac{(P_{bare}(\hat{v}) - N(\hat{v}))^2}{N_e(\hat{v})} \quad (36)$$

where $N_e(\hat{v})$ is defined as $N(\hat{v})$ when it is non zero and 1 otherwise.

Let us call $f_e(\hat{v})$ the ratio

$$\hat{f}_e(\hat{v}) = \frac{N_e(\hat{v})}{N_{tot} - L} \quad (37)$$

which is equal to $\hat{f}(\hat{v})$ (see (6)) when it is non zero but equal to $1/(N_{tot} - L)$ when $\hat{f}(\hat{v}) = 0$.

The final form of the criteria is

$$\chi^2 = (N_{tot} - L) \times \sum_{\hat{v}=\hat{v}_{min}-R}^{\hat{v}_{max}+R} \frac{(P(\hat{v}) - \hat{f}(\hat{v}))^2}{\hat{f}_e(\hat{v})}. \quad (38)$$

5.3 Fits with Padé $[0, 4]$. Estimation of the Parameters for all the Values of the Lags and Maturities

5.3.1 Normalized Padé $[0, 4]$

We have computed the best fit values of the parameters of a Padé $P^{[0,4]}(\hat{v})$, defined in (18), (20) and (19) but limited to non zero n_0, q_1, q_2 . The χ^2 (38) has been minimized with a normalized Padé ($n_0 = q_1/\pi$) (see (24)).

To that effect, we have used the Fortran IMSL DUNLSF minimization programme based on a modified Levenberg-Marquardt algorithm and a finite difference Jacobian. The best values of the parameters s, q_1 and q_2 (in respective units [%/year], [century], [century]²) and of their “one parameter standard error” $\Delta s, \Delta q_1$ and Δq_2 (in the same units) (see appendix B) have been computed for all values of the lag and of the maturity: altogether 210 different cases.

The results are given in Table 2 for all maturities but for lags $L = 1, 5, 10, 15, 20, 25, 30$ only, to keep a reasonable size for the table. The other values can be obtained from the authors.

Some remarks can be made at this point:

1. For all the cases, including those which are not reported in Table 2, except for the $[m] = [2], L = 1$ case, the hypothesis that the distribution is a Padé $[0, 4]$ cannot be rejected at the 5% confidence level. The exception is due to the fact that the experimental curve has an oscillatory behaviour around its maximum with three separate peaks. This is atypical as compared to all the other cases. We conjecture that this is due to a statistical fluctuation and does not correspond to a genuine and reproducible effect in the two years maturities.
2. The modulus of the shift parameter $|s|$ is always very small and can essentially be neglected. Indeed, it never exceeds a few [b.p./year] and in most cases, it is not statistically different from zero. This is a simple reflection of the fact that interest rates themselves follow cycles which come back to their original value.

This means that, in first approximation, the data are symmetrical under the $(v) \leftrightarrow (-v)$ exchange, as was foreseen and that the influence of s can be safely forgotten. Both the normalization and variance can, well within the experimental errors, be equated to the values obtained with $s = 0$.

3. For all values of the maturities, the two relevant parameters (and also n_0 which is by normalization a dependent parameters), q_1 and q_2 decrease very smoothly as a function of the lag, including the values of the lags which are not in Table 2. This is a simple reflection of the fact that as the lag becomes higher the distribution flattens. To a very good approximation, these decreasing functions are straight lines in a log-log plot. We will estimate the parameters of these straight lines in Section 5.4.
4. As a function of the maturity, the parameters first decrease then become higher showing a dip. Hence, the density function shows, first a tendency to flatten but becomes sharper when very long maturities are concerned. Experimental variations of the interest rates for the two and three years maturities seem to be somewhat less “smooth” than for the one year maturity. In turn, densities for maturities of 30 years are remarkably stable and strongly peaked around zero variations.
5. Even though as a function of lag and maturity together, no very simple pattern emerges, the smoothness of the parameters in the separate variables is a very welcome fact.

6. We have chosen to draw, in Figure 1, the experimental distribution together with the one given by the best Padé for the maturity $[m] = [1]$ and the lag $L = 1$ and in Figure 2 the case $[m] = [1]$ and $L = 30$. In both cases, the agreement between the two curves is rather good. The quality of the fit reflects the low value of the criteria equivalent to a goodness of fit of about one. All the analogous figures for the 210 cases show the same agreement. In one case only, the $[m] = [2]$, $L = 1$ case, the agreement is worse, reflecting the higher value of the criteria and a low goodness of fit. The related Figure 3 shows the reason for this fact. Instead of presenting one maximum, the experimental curve exhibits an oscillatory pattern around zero. The density of variation of interest rates is strongly peaked at $v = 2$, $v = 0$ and $v = -2$ instead of 0 as is the normal situation.
7. Though the Padé distributions do not have the properties of the Levy distributions, they enjoy with these distributions narrower peaks around their maximum and longer tails as compared to the Gaussian distributions (see below). Compared to the Levy distributions, they have the advantage of being expressed in a very simple rational form. As we have seen, many auxiliary computations can be done very easily and often analytically.

5.3.2 Normalized Padé $[0, 4]$ with Constrained Variance

The estimations of the parameters q_2 allows, in principle, to evaluate directly the variances (27). It turns out that the Padé variances do not always equate the values obtained directly from the data. As one can verify directly from Table 1 and Table 2, there is no systematic pattern in the deviations : for short maturities (1, 2 and 3 years) and large lags, the Padé variances underestimate the sample variances. The adequacy is good for intermediate maturities. For the longest maturity (30 years), the Padé variances overestimate slightly the sample variances.

These deviations are difficult to interpret. We have thus redone our computations using the parameter q_2 constrained to $-1/(Sample\ Variance)$. Even if the minimal values obtained for the criteria are higher than in the non-constrained cases, the value of the criteria remains fully acceptable except again in the $[m] = [2]$, $L = 1$ case which produces a too high χ^2 . Again, this exception is due to the fact that the data for $[m] = [2]$, $L = 1$ are atypical and that the experimental curve exhibits more than one maximum.

The parameters s are again rather small and, in most cases, not statistically different from zero. The values of the parameters s , q_1 and of χ^2 and of their standard errors are given in Table 3. We have restricted ourselves again to the lags $L = 1, 5, 10, 15, 20, 25, 30$ to keep the table of reasonable size. For the other lags, the parameter q_1 interpolates smoothly.

5.3.3 Comparison with the Gaussian Distribution

In order to compare the two cases corresponding to fits of Padé Approximants (with the constrained normalization and the variance constrained or not), we have performed the same computation for a Gaussian distribution of the form

$$G(v) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(v-s)^2}{2\sigma^2}} \quad (39)$$

with two free parameters σ and s for every lag and for every maturity. The Gaussian distributions lead to high χ^2 as compared to the Padé Approximants. For the non-constrained estimation, the hypothesis of a normal distribution is always rejected for lags up to five days and, in most cases, for the $[m] = [30]$ maturity. The no-rejection of the Gaussian model for some large lags is linked to a decrease in the leptokurticity with increasing time scale. These results are coherent with similar observations on the FX market (see [27]). If one constrains the standard deviation σ to its sample value, the normality is strongly rejected for all $[m], L$ cases.

As an example we have drawn in Figure 4 the curve for $L = 1$ and $[m] = [1]$ compared to the experimental points. This is obviously worse than the corresponding comparison for Padé $[0, 4]$ given in Figure 1. The Gaussian distribution is clearly wider for low v and drops more sharply for higher $|v|$ than the data points. The Padé curve is narrower for low v but drops less sharply for higher $|v|$ and hence fits the data much better. All the $[m], L$ cases present the same behaviour.

5.4 Fits with a Normalized Padé $[0, 4]$. Variation of the Parameters as a Function of the Lag for Different Maturities

The parameters $q_1^{[m]}(L)$ and $q_2^{[m]}(L)$ at their best value are simple functions of the lag L . More precisely, in a log-log plot, we can approximate them by a linear model of the form

$$\begin{aligned} \ln(q_1^{[m]}(L)) &= \lambda_1^{[m]} \ln\left(\frac{L}{L_0}\right) + \nu_1^{[m]} \\ \ln(-q_2^{[m]}(L)) &= \lambda_2^{[m]} \ln\left(\frac{L}{L_0}\right) + \nu_2^{[m]}. \end{aligned} \quad (40)$$

The justification of the arbitrary factor L_0 which will be taken equal to 15 is given in Appendix B where a precise discussion of the related standard errors is given.

The $q_i^{[m]}(L)$ are expressed in some units (see Appendix A). If these units are changed, $\lambda_i^{[m]}$ does not change while $\nu_i^{[m]}$ picks up, obviously, an additive constants (the logarithm of the ratio between the two units).

For the estimation of $\lambda_i^{[m]}$ and $\sigma_i^{[m]}$, we have to define new χ^2 ($C_i^{[m]}$) for every q_i and every maturity $[m]$. This takes into account the errors we have obtained on the q_i . They are

$$\begin{aligned} C_1^{[m]} &= \sum_{L=1}^{30} \frac{\left(e^{\nu_1^{[m]}(\frac{L}{L_0})\lambda_1^{[m]}} - q_1^{[m]}(L) \right)^2}{(\Delta q_1^{[m]}(L))^2} \\ C_2^{[m]} &= \sum_{L=1}^{30} \frac{\left(e^{\nu_2^{[m]}(\frac{L}{L_0})\lambda_2^{[m]}} + q_2^{[m]}(L) \right)^2}{(\Delta q_2^{[m]}(L))^2}. \end{aligned} \quad (41)$$

In Table 4, the values of the best $\lambda_i^{[m]}$ and $\nu_i^{[m]}$ minimizing $C_i^{[m]}$ and their one standard error $\Delta\lambda_i^{[m]}$ and $\Delta\nu_i^{[m]}$ are given for all maturities. In Figure 5 and Figure 6, the straight lines (40) are plotted for $[m] = [1]$ and compared to the data points. The fit is clearly excellent for $\ln(q_1^{[1]})$ and somewhat worse for $\ln(-q_2^{[1]})$.

The parameters λ_2 are related to the so-called “scaling laws” for the time dependence of the standard deviations

$$\sigma^{[m]}(L) = k^{[m]} L^{E^{[m]}}. \quad (42)$$

This behaviour is reported for most of the financial time series (see [25], [13], [27]) with an exponent typically close to but larger than 0.5, the value which should be observed for a Gaussian process. With our notations $E^{[m]} = -\lambda_2^{[m]}/2$. The absolute value of λ_2 , statistically greater than 1, lead to scaling exponents ranging from 0.52 to 0.54.

5.5 Discussion of Fits with Padé Approximants with Indices higher than $[0, 4]$

Since the Padé Approximants with indices $[0, 4]$ lead to such good fits to the data, it is tempting to see if these fits can even be made better by using higher indices. We have tried to do so and have come to the conclusion that $[0, 4]$ is close to the optimum.

1. First, the Padé Approximants $[M, N]$, when the condition of the existence of a finite variance is not satisfied (14), do not seem to be very probable. Those with $M=N$ clearly are not convenient as they cannot even be normalized.
2. When the condition of finite variance is satisfied (14), Padé with indices higher than $[0, 4]$ obviously produce somewhat better fits and hence lower values for the χ^2 . But this statement has to be qualified by considering in turn the different possibilities.

- (a) When $[0, 6]$ is used, the parameter q_3 , new with respect to our first choice, is significantly different from zero in few cases only. The χ^2 is not improved in most cases. This means that the $[0, 4]$ approximation is already able to cope with the decrease of the experimental density function. A v^{-4} is sufficient to achieve it. Our careful, but, may be, slightly prejudiced analysis of the data even seems to point to the fact that the v^{-4} decrease is the fastest which can be allowed by the data. Indeed, the fit to the variance ($-1/q_2$) coming out of the minimization already tends to come out too small as compared to the raw variance of the data, for maturities up to three years. Remember that this raw variance depends crucially on the end points of the experimental distribution and hence is not determined very precisely. For $P^{[0,6]}$, the moment of fourth degree (numerator of the kurtosis) then becomes finite which may be interesting theoretically though to repeat, we believe that the data point in the infinite kurtosis direction. The same findings were achieved on the FX rates in [7].
- (b) We have also tried to fit the experimental data with the other Padé's ($[0, 8]$, $[2, 8]$, $[4, 8]$ and $[2, 6]$). In the majority of the cases, no significant improvement of the χ^2 is obtained. In some cases, the minimization algorithm fails $[2, 6]$. In a few cases our minimizations lead to a smaller value for the criteria. But, the price to pay is that the Padé curve starts to try to fit small oscillations in the experimental curves $[0, 8]$, $[2, 8]$ and $[4, 8]$. These oscillations are obviously due to the finiteness of the sample and do not very likely correspond to intrinsic structures. Moreover, the minimum obtained are usually very sensitive to the starting values and the minimal parameters do not exhibit the stability they show in the $[0, 4]$ case.

6 Conclusions

In this paper, we have presented and discussed phenomenological fits of the daily variations of the term structure of interest rates (taking as an example those published by the Board of Governors of the Federal Reserve System [11]). We have shown that simple Padé Approximants, which are theoretically well suited for approximating in a rather smooth way many classes of functions, fit the experimental results amazingly well.

The best values of the parameters defining the Padé Approximants vary rather smoothly as a function of the lag and of the maturity. In particular, as a function of the lag, in a log-log plot the parameters are very well represented by straight lines. This can be related to scaling laws reported for other financial time series.

We have also shown that, the simplest Padé Approximant $P^{[0,4]}$, which depends essentially on only two meaningful parameters, has enough richness to

represent the data faithfully and smoothly. The extension to higher Padé Approximants (for example to $P^{[0,6]}$) is by no means necessary. The Padé Approximants have then a tendency to mimic small oscillations in the data. These oscillations are clearly the results of the finiteness of the sample and do not seem to be related to any “real” underlying structure.

Since Padé Approximants are simple rational functions, they can be easily written, tabulated, and, once the best parameters have been obtained, used for further applications. We hope to come back to this idea in the near future and show how these explicit results can be used for estimating risks related to extreme events.

A Dimensions of the Padé Parameters

A short appendix about units and dimensions is useful here. Interest rates have the dimension of the inverse of time, $[[t]]^{-1}$, are usually given in [%/year] with exactly two meaningful digits and hence are expressed in [b.p./year] as an integer number.

- This makes [b.p./year] = 10^{-4} /year a very natural unit to use both for the interest rates but even more for the variation in the interest rates. It gives a natural binning for the discretized distributions. The inverse unit of [b.p./year] is [year/b.p.] which is the time scale of 10^4 years. This is the unit we have consistently used in the main body of the paper.
- In the tables and in the figures, the different parameters as well as the means and the variances are expressed as powers of the more conventional unit [%/year] and of its inverse [year/%] which is a [century]. They are units of well adapted sizes.
- In the following table, we quote the dimension $[[x]]$ of the quantities x we have been using, the units $[x]$ we have been using in the main part of the text and the units we have used in the tables.

Table of Dimensions and Units

variable	[[dimension]]	[unit] used in the body of the paper	[unit] used in the tables	restriction
v	$[[t]]^{-1}$	[b.p./year]	[%/year]	
s	$[[t]]^{-1}$	[b.p./year]	[%/year]	
f	$[[t]]$	[year/b.p.]	[century]	
P	$[[t]]$	[year/b.p.]	[century]	
n_i	$[[t]]^{1+i}$	[year/b.p.] $^{i+1}$	[century] $^{i+1}$	
u_0	$[[t]]^{\frac{1}{2}}$	[year/b.p.] $^{\frac{1}{2}}$	[century] $^{\frac{1}{2}}$	
u_i	$[[t]]^i$	[year/b.p.] i	[century] i	for $i = 1, 2$
q_i	$[[t]]^i$	[year/b.p.] i	[century] i	
d_i	$[[t]]^i$	[year/b.p.] i	[century] i	
Mean	$[[t]]^{-1}$	[b.p./year]	[%/year]	
Variance	$[[t]]^{-2}$	[b.p./year] 2	[%/year] 2	
Skewness	$[[t]]^0$			
Kurtosis	$[[t]]^0$			
χ^2	$[[t]]^0$			

- It is easy to pass from variables expressed in one unit to the variables expressed in the other unit by using the correspondence

$$\begin{aligned}
[\text{b.p./year}] &= \frac{[\%/year]}{100} \\
[\text{year/b.p.}] &= 100 \times [\text{year}/\%] = 100 \times \text{century} .
\end{aligned} \tag{43}$$

- For example, the value of q_i expressed in units $[\text{year/b.p.}]^i$ becomes $100^i \times q_i$ in the more customary unit $[\text{year}/\%]^i$ while n_i becomes $100^{i+1} \times n_i$. The Padé density $P(v, n_i, q_i)$ (where we have written explicitly the parameters (18),(19)) in units $[\text{year/b.p.}]$ for v expressed in $[\text{b.p./year}]$ is related to the density $P^{new}(v)$ where P^{new} is expressed in $[\text{century}]$ in terms of v expressed in $[\%/year]$ by

$$P^{new}(v) = P(v; 100^{i+1}n_i, 100^i q_i) . \tag{44}$$

B Discussion of the Errors

A short discussion on the determination of the errors is worthwhile giving. The χ^2 ((38) or (41)), depend on the Padé or on the λ, ν parameters. Let us call these parameters generically $p = \{p_1, p_2, \dots\}$. The best values of the parameters p^{best} are obtained by minimizing the criteria $\chi^2(p^{best}) = \chi_{min}^2$. The confidence regions in one parameter with confidence level $p = 68.3\%$ are obtained for the contour

$$\Delta\chi^2 \equiv \chi^2(p) - \chi_{min}^2 = 1 . \tag{45}$$

For errors on two parameters jointly, the contour is given by

$$\Delta\chi^2 = 2.3. \quad (46)$$

From the computation of the matrix H of the second derivatives (the Hessian) of the Criteria χ^2 at its minimum and from the Taylor expansion of $\chi^2(p)$ (to second order in the parameters) one finds

$$\Delta\chi^2 = \frac{1}{2}H_{ij}(p_i - p_i^{best})(p_j - p_j^{best}). \quad (47)$$

- If this ellipsoid has its principal axis reasonably aligned along the parameter axis (which is the case for the parameters we have chosen as can be seen on Figures 7, 8 and 9) one can define the “one parameter standard error” Δp_i , for a given parameter p_i as

$$\Delta p_i \equiv |p_i - p_i^{best}| = \sqrt{\frac{2}{H_{ii}}}. \quad (48)$$

Notice that the classical interpretation of standard errors is only valid if the measurements errors are normally distributed. If they are not, a suitable contour of constant χ^2 ((45), (46)) should be used as boundary of the confidence region (see [29]) for a more precise discussion).

- If this ellipsoid does not have its principal axis reasonably aligned along the axis, the “one parameter error” is obtained from the maximal M_i and minimal m_i values of the fixed parameter p_i in the region $\Delta\chi^2 \leq 1$ as

$$\Delta p_i = \frac{M_i - m_i}{2}. \quad (49)$$

It should be remarked that for (41), the (obvious) value $L_0 = 15$ is close to the optimal value which makes the principal axis of the ellipse parallel to the parameter axis.

As examples, the curves corresponding to $\Delta\chi^2 = 1$ and $\Delta\chi^2 = 2.3$ have been drawn for the parameters q_1, q_2 in the $[m] = [1]$, $L = 1$ case in Figure 7, for the parameters λ_1 and ν_1 for $[m] = [1]$ in Figure 8 and for the parameters λ_2 and ν_2 for $[m] = [1]$ in Figure 9.

C Analyticity of the Denominator

The Normalization (21), the Variance (25) and the Kurtosis (28) have been computed analytically from the form we have given for the denominator of the Padé Approximant (19). There is a restriction for these formulas to be valid, namely

that the poles of $Q^n(v)$ be all situated in, say, the upper-half plane of the complexified variable v . All the poles of $Q^{n*}(v)$ are then situated in the lower half-plane. If the poles have complex values $v_j = s + a_j + ib_j$, $j = 1, \dots, 4$, with $b_j > 0$, $Q^n(v)$ is written

$$Q^n(v) = \prod_{j=1}^n (1 + i \frac{v - s}{v_j}). \quad (50)$$

Expanding the products and taking account of the positivity of the b_j , it is easy to see this implies for Q^3 the necessary conditions are

$$\begin{aligned} q_1 &> 0, & q_2 &< 0 \\ q_3 &\leq 0, & q_3 - q_1 q_2 &\geq 0. \end{aligned}$$

We expect that these conditions, for Q^4 , are supplemented by the conditions

$$\begin{aligned} q_4 &\geq 0, & q_2 q_3 - q_1 q_4 &\geq 0 \\ q_1 q_2 q_3 - q_1^2 q_4 - q_3^2 &\geq 0 \end{aligned}$$

Moreover, written in terms of the poles, we have verified that all the analytical forms of the Normalization, of the Variance and the Kurtosis lead to positive values and hence are fully meaningful.

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Table Captions

- Table 1

Table of Univariate Statistics for the daily changes of the American Spot Interest Rates (expressed in the relevant [%/year] unit) between February 15, 1977 and August 4, 1997. The statistics are given for all available maturities $[m] = [1], [2], [3], [5], [7], [10], [30]$ and for the representative subset of lags $L = 1, 5, 10, 15, 20, 25$ and $L = 30$. The quantities are expressed in the relevant [%/year] unit. See appendix A.

- Table 2

The optimal values of the parameters q_1 , q_2 and s appearing in the denominator of the Padé $P^{[0,4]}$ together with their “errors” Δq_1 , Δq_2 and Δs (for the units chosen, see appendix A, for the interpretation of the errors, see appendix B). The parameters n_0 are constrained by the normalization. These optimal parameters are given for all available maturities $[m] = [1], [2], [3], [5], [7], [10], [30]$ and for the representative subset of lags $L = 1, 5, 10, 15, 20, 25$ and $L = 30$. The parameters for the other values of the lag have been computed but are not given to limit the size of the table. They vary smoothly and have been taken into account to estimate the parameters λ_i and ν_i appearing in Table 4. The minimal value of the Criteria χ^2 (a pure number) is also given. For all the cases, including those which are not reported in the table, except for the $[m] = [2]$, $L = 1$ case whose χ^2 is marked with a \star in the table, the hypothesis that the distribution is a Padé $[0, 4]$ cannot be rejected at the 5% confidence level.

- Table 3

The optimal values of the parameters q_1 and s appearing in the denominator of the Padé $P^{[0,4]}$ together with their errors Δq_1 , Δs (for the units chosen, see appendix A, for the interpretation of the errors, see appendix B). In this table, the values of n_0 are constrained by the normalization and the values of q_2 by the experimental variance. These parameters are given for all available maturities $[m] = [1], [2], [3], [5], [7], [10], [30]$ and for the representative subset of lags $L = 1, 5, 10, 15, 20, 25, 30$. The parameters for the other values of the lag have been computed but are not given to limit the size of the table. They vary smoothly. The minimal value of the Criteria χ^2 (a pure number) is also given. For all the cases, including those which are not reported in the table, except for the $[m] = [2]$, $L = 1$ case whose χ^2 is marked with a \star in the table, the hypothesis that the distribution is a Padé $[0, 4]$ cannot be rejected at the 5% confidence level.

- Table 4

The optimal values of the λ_{q_1} , ν_{q_1} , λ_{q_2} , ν_{q_2} parameters and their standard errors $\Delta\lambda_{q_1}$, $\Delta\nu_{q_1}$, $\Delta\lambda_{q_2}$, $\Delta\nu_{q_2}$ as functions of the maturity (for the interpretation of the errors, see appendix B).

Figure Captions

- Figure 1a

The experimental distribution compared with the one given by the best Padé $[0, 4]$ for maturity $[m] = [1]$ and lag $L = 1$.

- Figure 1b

The difference between the experimental distribution and the distribution obtained by the best Padé $[0, 4]$ for maturity $[m] = [1]$ and lag $L = 1$. Comparing this figure to Figure 1a, one sees that, in absolute value, the deviation between the data and the approximation is larger in the central part of the distribution but with no systematic bias.

- Figure 1c

The ratio R between the experimental density and the best Padé $[0, 4]$ density for maturity $[m] = [1]$ and lag $L = 1$. This figure zooms the tails as compared to the central values. One sees that, in relative size, the central part is very well reproduced. For absolute values of v higher than 0.4, the scarcity of the data is reflected by the appearance in the plot of the horizontal axis and of curves rising as v^4 . These correspond to the rare events $N(\hat{v}) = 0, 1, 2, 3$ (see (3)) in the tails of the distribution of variations. In this region, the Padé distribution averages the data.

- Figure 2

The experimental distribution compared with the one given by the best Padé $[0, 4]$ for the maturity $[m] = [1]$ and the lag $L = 30$.

- Figure 3

The exceptional case of the experimental distribution compared with the one given by the best Padé $[0, 4]$ for the maturity $[m] = [2]$ and the lag $L = 1$. The experimental points exhibit three very large oscillations in the region around $v = 0$ showed in the figure. We have highlighted this fact by arbitrarily drawing the dotted line. These oscillations may be of statistical nature as they do not show up in any of the 209 other cases that we have studied.

- Figure 4a

The experimental distribution compared to the Gaussian distribution for the maturity $[m] = [1]$ and the lag $L = 1$. The experimental distribution is more strongly peaked than the Gaussian distribution which itself has too small tails.

- Figure 4b

The difference between the experimental distribution and the Gaussian distribution for the maturity $[m] = [1]$ and the lag $L = 1$. This figure is worth comparing with Figure 1b. One sees that, in absolute value, the deviation between the data and the approximation is large in the central part of the distribution but also shows a systematic bias in the region $0.2 \leq |v| \leq 0.4$.

- Figure 4c

The ratio R between the experimental density and the best Gaussian density for maturity $[m] = [1]$ and lag $L = 1$. This figure zooms the tails as compared to the central values. One sees that, in relative size, neither the central part nor the tails are very well reproduced. For absolute values of v higher than 0.2, R is seen to shoot up extremely fast. Hence, the tails are very badly fitted by a Gaussian.

- Figure 5

The straight line $\ln(q_1)$ (see (40)) as a function of $\ln(L/L_0)$ compared with the data points for $[m] = [1]$. The parameters are obtained from the best fit with a normalized Padé [0.4].

- Figure 6

The straight line $\ln(-q_2)$ (see (40)) as a function of $\ln(L/L_0)$ compared with the data points for $[m] = [1]$.

- Figure 7

The ellipses corresponding to $\Delta\chi^2 = 1$ and $\Delta\chi^2 = 2.3$ for the errors in the parameters q_1, q_2 appearing in the $[m] = [1], L = 1$ case. The estimates of q_1 and q_2 are found in Table 2.

- Figure 8

The ellipses corresponding to $\Delta\chi^2 = 1$ and $\Delta\chi^2 = 2.3$ for the errors in the parameters λ_1, ν_1 appearing in the linear model of $\ln(q_1)$ as a function of $\ln(L/L_0)$ in the $[m] = [1]$ case. The estimates of λ_1 and ν_1 are found in Table 4.

- Figure 9

The ellipses corresponding to $\Delta\chi^2 = 1$ and $\Delta\chi^2 = 2.3$ for the errors in the parameters λ_2, ν_2 appearing in the linear model of $\ln(-q_2)$ as a function of $\ln(L/L_0)$ in the $[m] = [1]$ case. The estimates of λ_2 and ν_2 are found in Table 4.

Table 1
Table of Univariate Statistics

	L	1	5	10	15	20	25	30
[m]=[1]	Mean	0.0000	0.0001	0.0000	0.0001	0.0001	0.0002	0.0004
	Variance	0.0143	0.0860	0.1966	0.3326	0.4836	0.6446	0.8136
	Skewness	-0.1541	-0.8223	-1.0587	-1.0710	-1.1273	-1.1413	-1.1594
	Kurtosis	14.5664	10.7279	10.4445	10.9208	10.9326	10.9088	10.8980
	v_{min}	-1.0800	-2.2700	-3.0600	-4.0700	-5.4500	-6.3100	-6.9100
	v_{max}	1.1000	2.0600	2.4700	3.1400	3.2000	3.7200	4.0100
[m]=[2]	Mean	0.0000	-0.0002	-0.0005	-0.0007	-0.0009	-0.0009	-0.0010
	Variance	0.0113	0.0702	0.1594	0.2663	0.3836	0.5109	0.6456
	Skewness	-0.3648	-0.7398	-0.8722	-0.8171	-0.8547	-0.8600	-0.8741
	Kurtosis	12.4522	9.7384	9.1389	9.2421	8.8631	8.5522	8.3537
	v_{min}	-0.8400	-2.0800	-2.7900	-3.3700	-4.4900	-5.2300	-5.9600
	v_{max}	0.8900	1.9900	2.6000	2.8700	3.0900	3.3900	3.7100
[m]=[3]	Mean	-0.0001	-0.0004	-0.0010	-0.0015	-0.0019	-0.0022	-0.0025
	Variance	0.0102	0.0622	0.1374	0.2256	0.3225	0.4285	0.5421
	Skewness	-0.1628	-0.4432	-0.5977	-0.5000	-0.5254	-0.5487	-0.5928
	Kurtosis	10.4136	7.7748	7.6880	7.4571	6.7799	6.4504	6.3316
	v_{min}	-0.7900	-1.5700	-2.8300	-3.0500	-3.5800	-4.3200	-5.1200
	v_{max}	0.9200	1.9900	2.6000	3.0400	3.3200	3.6000	3.6500
[m]=[5]	Mean	-0.0001	-0.0007	-0.0016	-0.0024	-0.0032	-0.0038	-0.0044
	Variance	0.0091	0.0544	0.1182	0.1901	0.2661	0.3491	0.4409
	Skewness	-0.3064	-0.3938	-0.4482	-0.3617	-0.3448	-0.3109	-0.3025
	Kurtosis	8.7216	6.2382	5.9828	5.9650	5.5396	4.8740	4.4440
	v_{min}	-0.7700	-1.5400	-2.3900	-2.5800	-3.0700	-3.6000	-4.3200
	v_{max}	0.7200	1.6400	2.3600	2.8200	3.0800	3.3800	3.5100
[m]=[7]	Mean	-0.0002	-0.0010	-0.0021	-0.0031	-0.0040	-0.0048	-0.0056
	Variance	0.0085	0.0491	0.1044	0.1651	0.2287	0.2981	0.3749
	Skewness	-0.3066	-0.3907	-0.4981	-0.3602	-0.3019	-0.2683	-0.2637
	Kurtosis	8.1780	5.3213	5.4510	5.2149	4.6201	3.9410	3.5034
	v_{min}	-0.7800	-1.3600	-2.4000	-2.5500	-2.9400	-3.1100	-3.6200
	v_{max}	0.7000	1.5300	2.0300	2.6000	2.8000	3.1100	3.2600
[m]=[10]	Mean	-0.0002	-0.0012	-0.0025	-0.0037	-0.0049	-0.0060	-0.0070
	Variance	0.0076	0.0440	0.0928	0.1449	0.1984	0.2574	0.3224
	Skewness	-0.2817	-0.5431	-0.6719	-0.4999	-0.3918	-0.3246	-0.3022
	Kurtosis	6.9688	5.4452	5.7715	5.1414	4.1992	3.4414	3.0903
	v_{min}	-0.7500	-1.3600	-2.3500	-2.6000	-2.6200	-2.8400	-3.3700
	v_{max}	0.6500	1.3600	1.8500	2.3600	2.5500	2.8300	2.9700
[m]=[30]	Mean	-0.0002	-0.0013	-0.0027	-0.0039	-0.0052	-0.0062	-0.0073
	Variance	0.0062	0.0349	0.0723	0.1114	0.1518	0.1972	0.2475
	Skewness	-0.2245	-0.4463	-0.4792	-0.3334	-0.2077	-0.1437	-0.1078
	Kurtosis	6.8619	4.5314	4.0574	3.7200	3.1661	2.6781	2.3830
	v_{min}	-0.7600	-1.3100	-1.8800	-2.1600	-2.3800	-2.2000	-2.4800
	v_{max}	0.5000	0.9500	1.3800	1.6500	2.0800	2.3700	2.4900

Table 2
Parameters for the Normalized Padé Approximant $[0, 4]$

		$L=1$	$L=5$	$L=10$	$L=15$	$L=20$	$L=25$	$L=30$
[m]=[1]	χ^2	140.59	310.90	446.62	508.29	668.08	660.21	665.98
	q_1	25.253	8.633	5.655	4.474	3.648	3.162	2.885
	Δq_1	0.518	0.171	0.110	0.085	0.067	0.062	0.056
	q_2	-70.861	-16.802	-9.145	-5.472	-4.053	-2.867	-2.125
	Δq_2	4.987	0.814	0.369	0.257	0.177	0.130	0.110
	s	-0.00071	0.00549	0.00968	0.01143	0.02594	0.01579	0.01005
	Δs	0.00078	0.00227	0.00331	0.00419	0.00495	0.00584	0.00654
[m]=[2]	χ^2	367.28*	275.87	361.33	453.41	527.96	521.65	534.96
	q_1	24.073	8.022	5.139	3.940	3.161	2.749	2.511
	Δq_1	0.475	0.163	0.100	0.074	0.058	0.052	0.047
	q_2	-84.096	-17.733	-9.349	-5.917	-4.286	-3.023	-2.204
	Δq_2	4.744	0.683	0.338	0.227	0.151	0.109	0.086
	s	-0.00009	0.00506	0.01052	0.01278	0.02027	0.01102	-0.00135
	Δs	0.00081	0.00240	0.00349	0.00434	0.00512	0.00616	0.00709
[m]=[3]	χ^2	123.82	272.21	324.87	372.72	504.58	513.08	514.94
	q_1	24.291	7.850	5.099	3.8345	3.158	2.765	2.528
	Δq_1	0.497	0.161	0.100	0.073	0.057	0.051	0.047
	q_2	-89.184	-18.659	-8.811	-5.438	-4.091	-2.894	-2.132
	Δq_2	4.989	0.674	0.309	0.206	0.153	0.107	0.089
	s	0.00016	0.00512	0.00563	0.00477	0.01984	0.01253	0.001
	Δs	0.00083	0.00241	0.00359	0.00458	0.00523	0.00625	0.00711
[m]=[5]	χ^2	130.61	205.42	280.57	330.08	428.37	500.15	501.82
	q_1	24.225	7.976	5.295	4.099	3.387	2.988	2.758
	Δq_1	0.510	0.159	0.103	0.079	0.063	0.056	0.051
	q_2	-96.702	-18.222	-9.425	-5.844	-4.391	-3.092	-2.313
	Δq_2	4.700	0.724	0.356	0.233	0.166	0.120	0.103
	s	0.00006	0.00206	0.00659	0.00236	0.00124	-0.00026	-0.00424
	Δs	0.00082	0.00240	0.00346	0.00439	0.00509	0.00599	0.00669
[m]=[7]	χ^2	124.42	170.78	260.73	335.07	339.22	367.99	516.64
	q_1	23.745	8.038	5.602	4.355	3.546	3.103	2.945
	Δq_1	0.492	0.161	0.111	0.085	0.066	0.059	0.053
	q_2	-100.715	-19.25	-9.537	-6.209	-4.393	-3.324	-2.860
	Δq_2	4.739	0.751	0.375	0.244	0.174	0.131	0.132
	s	-0.0004	0.00177	0.00447	0.00245	-0.00041	0.00133	-0.00934
	Δs	0.00084	0.00238	0.00338	0.00423	0.00506	0.00582	0.00607
[m]=[10]	χ^2	137.93	175.08	229.76	319.02	377.43	321.47	499.17
	q_1	24.505	8.527	5.815	4.585	3.824	3.291	3.126
	Δq_1	0.505	0.169	0.113	0.091	0.073	0.064	0.058
	q_2	-109.545	-21.157	-10.172	-7.088	-5.026	-3.594	-3.114
	Δq_2	5.035	0.825	0.405	0.271	0.193	0.143	0.139
	s	-0.00074	0.00204	0.00706	0.00224	0.0033	0.00055	0.00475
	Δs	0.00081	0.00225	0.00326	0.00399	0.00474	0.0056	0.00581
[m]=[30]	χ^2	102.78	166.29	238.66	280.98	337.20	299.56	464.49
	q_1	27.439	9.848	6.878	5.481	4.548	3.912	3.656
	Δq_1	0.579	0.196	0.135	0.107	0.087	0.075	0.070
	q_2	-128.242	-25.041	-12.123	-7.921	-5.759	-4.234	-3.398
	Δq_2	5.661	1.028	0.518	0.340	0.257	0.189	0.167
	s	-0.00051	0.00134	0.00463	0.00145	0.0018	0.00408	-0.00445
	Δs	0.00073	0.00199	0.00283	0.00353	0.00419	0.0049	0.00525

Table 3
Parameters for the Normalized Padé Approximant $[0, 4]$ with Constrained Variance

		$L=1$	$L=5$	$L=10$	$L=15$	$L=20$	$L=25$	$L=30$
[m]=[1]	χ^2	140.62	353.13	573.23	590.96	765.57	736.57	719.84
	q_1	25.262	8.965	6.130	4.807	3.968	3.425	3.049
	Δq_1	0.517	0.172	0.112	0.085	0.067	0.061	0.055
	s	-0.00071	0.00421	0.00633	0.00564	0.02711	0.00749	0.00832
	Δs	0.00078	0.00231	0.00346	0.00442	0.00541	0.00612	0.00682
[m]=[2]	χ^2	368.13*	305.13	457.03	552.25	660.94	622.24	585.35
	q_1	24.056	8.163	5.403	4.190	3.395	2.928	2.619
	Δq_1	0.475	0.166	0.103	0.075	0.059	0.054	0.049
	s	-0.00008	0.00456	0.01033	0.01152	0.01581	0.00845	-0.00263
	Δs	0.00081	0.00241	0.00363	0.00469	0.00568	0.00659	0.00740
[m]=[3]	χ^2	126.50	288.11	352.29	398.49	550.75	542.31	525.85
	q_1	24.260	7.928	5.163	3.905	3.244	2.816	2.550
	Δq_1	0.497	0.163	0.102	0.073	0.057	0.058	0.052
	s	0.00013	0.00549	0.00661	0.00538	0.01881	0.01162	0.00225
	Δs	0.00082	0.00242	0.00363	0.00471	0.00552	0.00644	0.00727
[m]=[5]	χ^2	137.99	205.47	288.40	336.55	443.68	503.88	502.02
	q_1	24.276	7.975	5.328	4.135	3.429	2.998	2.760
	Δq_1	0.509	0.159	0.103	0.079	0.063	0.056	0.051
	s	0.00005	0.00203	0.00689	0.00251	0.00088	0.00006	-0.00402
	Δs	0.00081	0.00240	0.00349	0.00444	0.00520	0.00604	0.00671
[m]=[7]	χ^2	136.33	172.98	260.75	335.45	339.23	368.04	516.76
	q_1	23.817	8.032	5.601	4.360	3.547	3.102	2.958
	Δq_1	0.490	0.161	0.111	0.085	0.066	0.059	0.053
	s	-0.00046	0.00171	0.00447	0.00265	-0.00040	0.00123	-0.00896
	Δs	0.00083	0.00237	0.00338	0.00424	0.00506	0.00581	0.00614
[m]=[10]	χ^2	153.87	178.63	231.94	319.50	377.44	325.44	499.18
	q_1	24.630	8.527	5.810	4.592	3.823	3.286	3.127
	Δq_1	0.505	0.169	0.113	0.091	0.073	0.064	0.058
	s	-0.00077	0.00189	0.00722	0.00234	0.00328	0.00012	0.00480
	Δs	0.00080	0.00224	0.00324	0.00400	0.00474	0.00556	0.00581
[m]=[30]	χ^2	126.60	177.63	248.66	289.85	346.74	317.15	478.07
	q_1	27.836	9.876	6.888	5.486	4.536	3.909	3.638
	Δq_1	0.578	0.195	0.135	0.107	0.087	0.075	0.070
	s	-0.00060	0.00109	0.00430	0.00123	0.00167	0.00275	-0.00657
	Δs	0.00071	0.00197	0.00280	0.00349	0.00412	0.00481	0.00516

Table 4
Parameters of the Padé $[0, 4]$ as a Function of the Lag for Different Maturities

	$[m] = [1]$	$[m] = [2]$	$[m] = [3]$	$[m] = [5]$	$[m] = [7]$	$[m] = [10]$	$[m] = [30]$
λ_{q_1}	-0.663	-0.666	-0.660	-0.623	-0.604	-0.594	-0.582
$\Delta\lambda_{q_1}$	0.004	0.004	0.004	0.004	0.004	0.004	0.004
ν_{q_1}	1.480	1.363	1.350	1.411	1.459	1.513	1.676
$\Delta\nu_{q_1}$	0.004	0.004	0.004	0.004	0.004	0.004	0.004
λ_{q_2}	-1.054	-1.070	-1.085	-1.076	-1.054	-1.046	-1.052
$\Delta\lambda_{q_2}$	0.010	0.009	0.009	0.008	0.009	0.008	0.009
ν_{q_2}	1.647	1.684	1.670	1.737	1.803	1.907	2.042
$\Delta\nu_{q_2}$	0.008	0.007	0.007	0.007	0.007	0.007	0.008

Figure 1a

Pade [0,4] Density Compared to the Data

$$[m] = [1] , L = 1$$

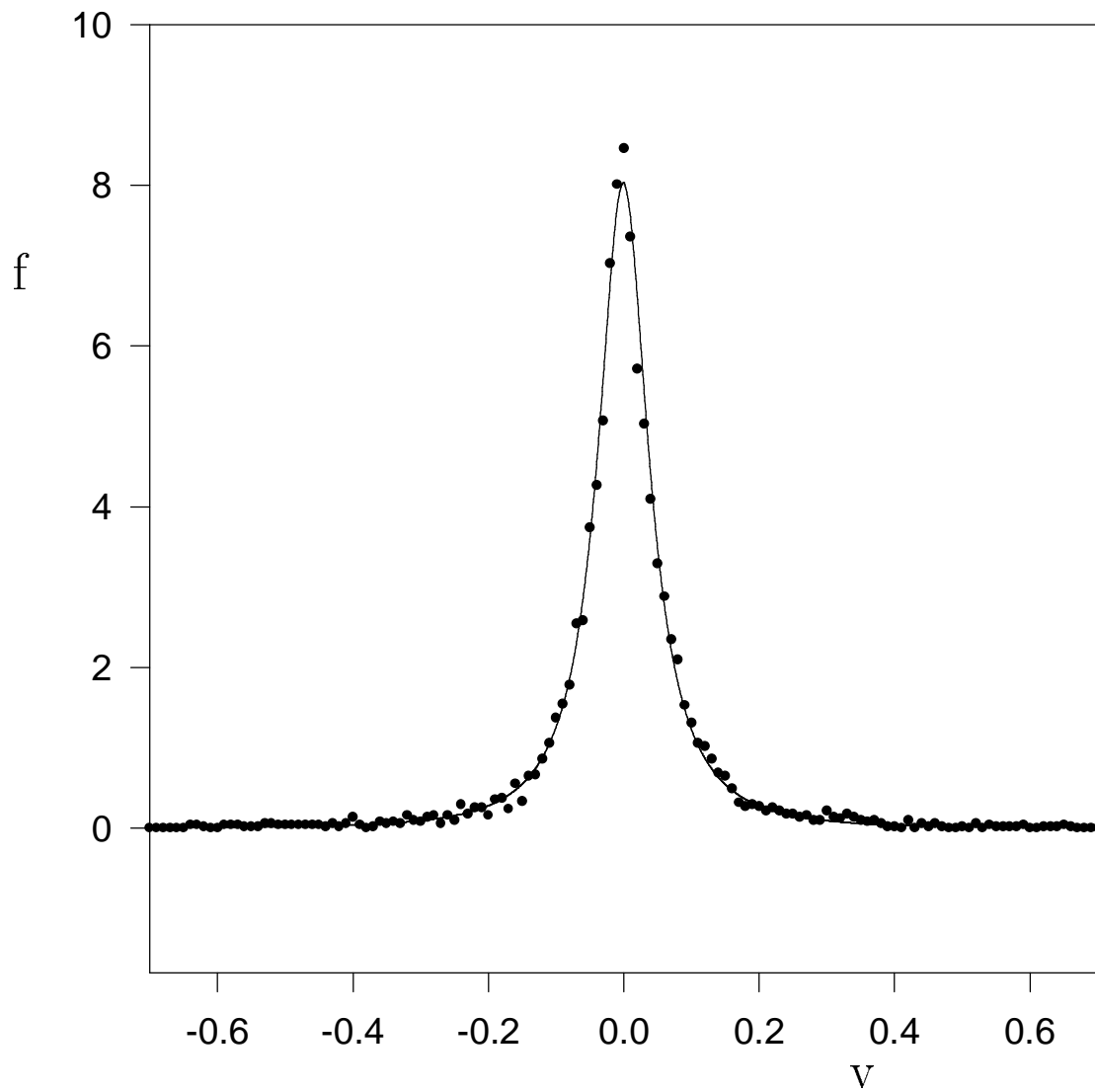


Figure 1b

Density Difference between
the Data and Pade [0,4]

$$[m] = [1], L = 1$$

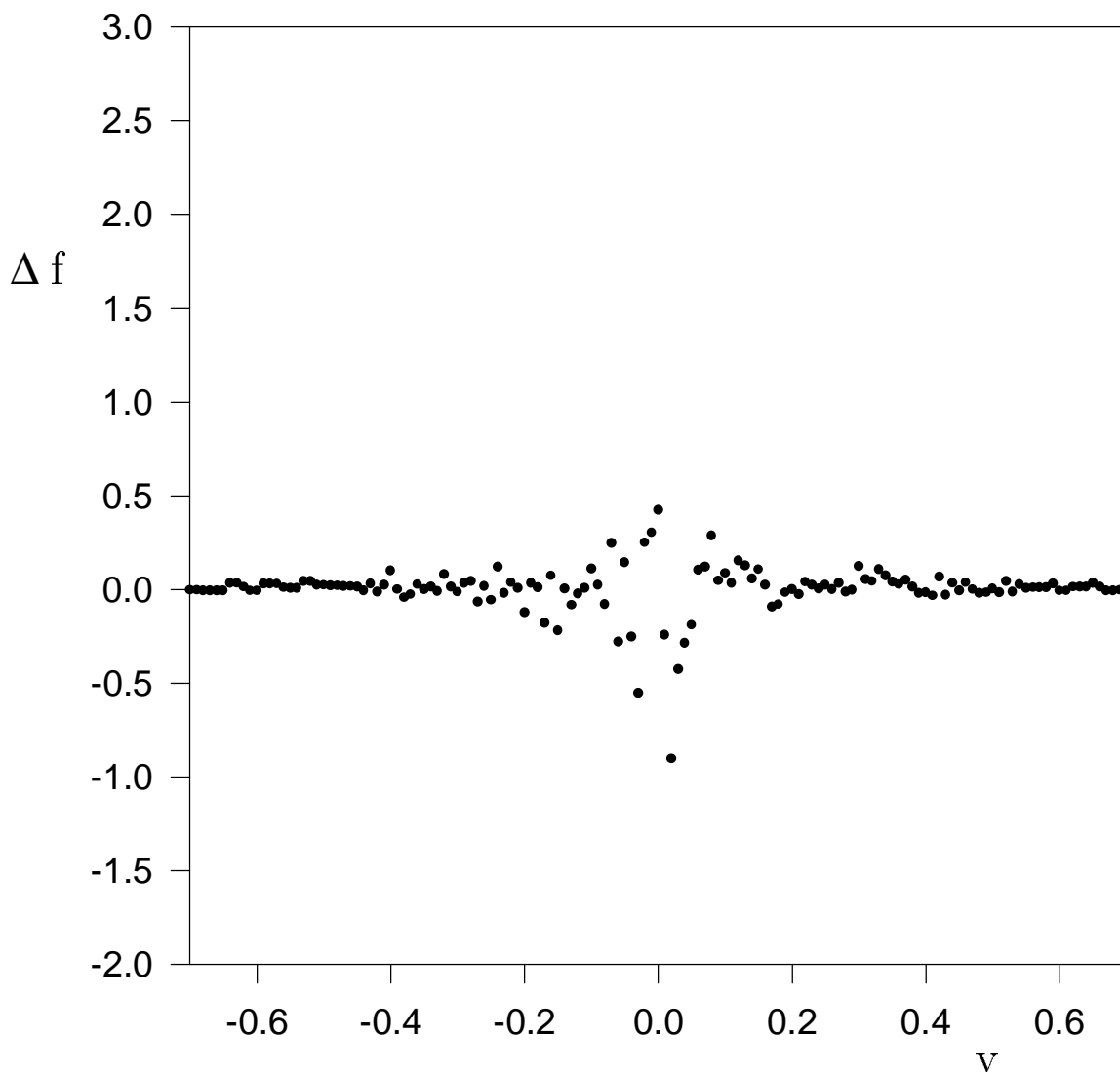


Figure 1c

Ratio R of the Data to the Pade [0,4]

Densities for $[m] = [1]$, $L = 1$

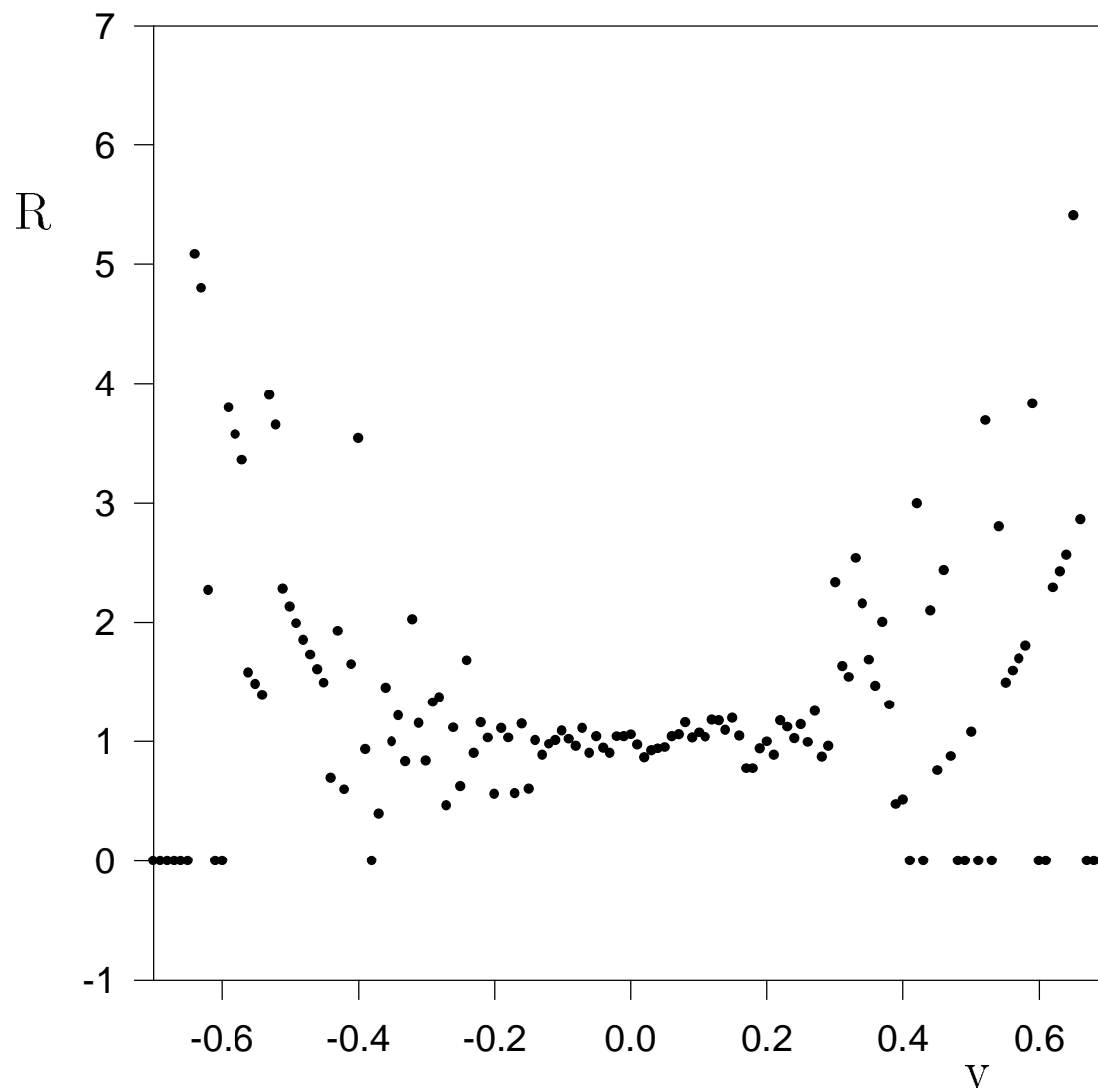


Figure 2

Pade [0,4] Density Compared to the Data

$$[m] = [1] , L = 30$$

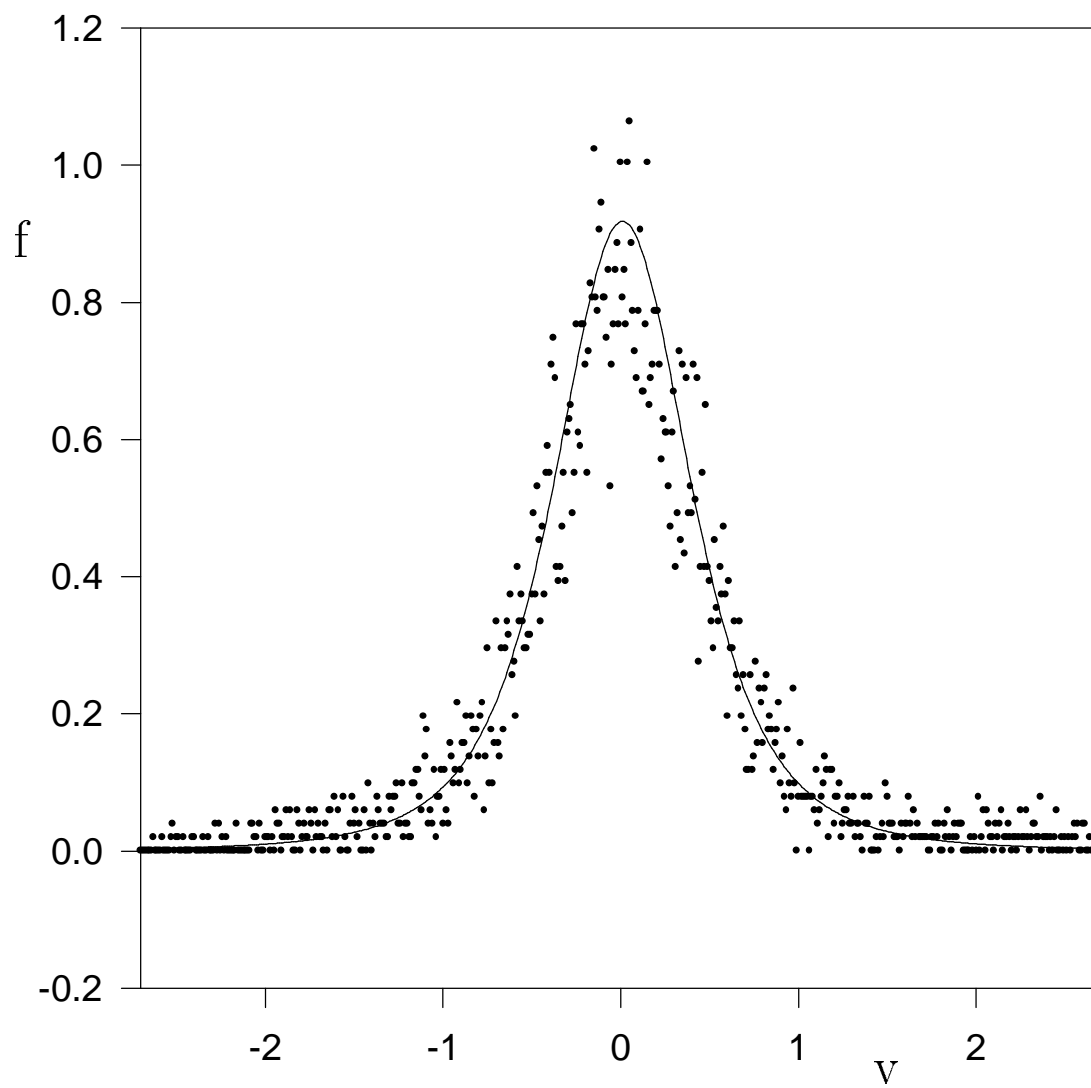


Figure 3

Pade [0,4] Density Compared to Data

$$[m] = [2] , L = 1$$

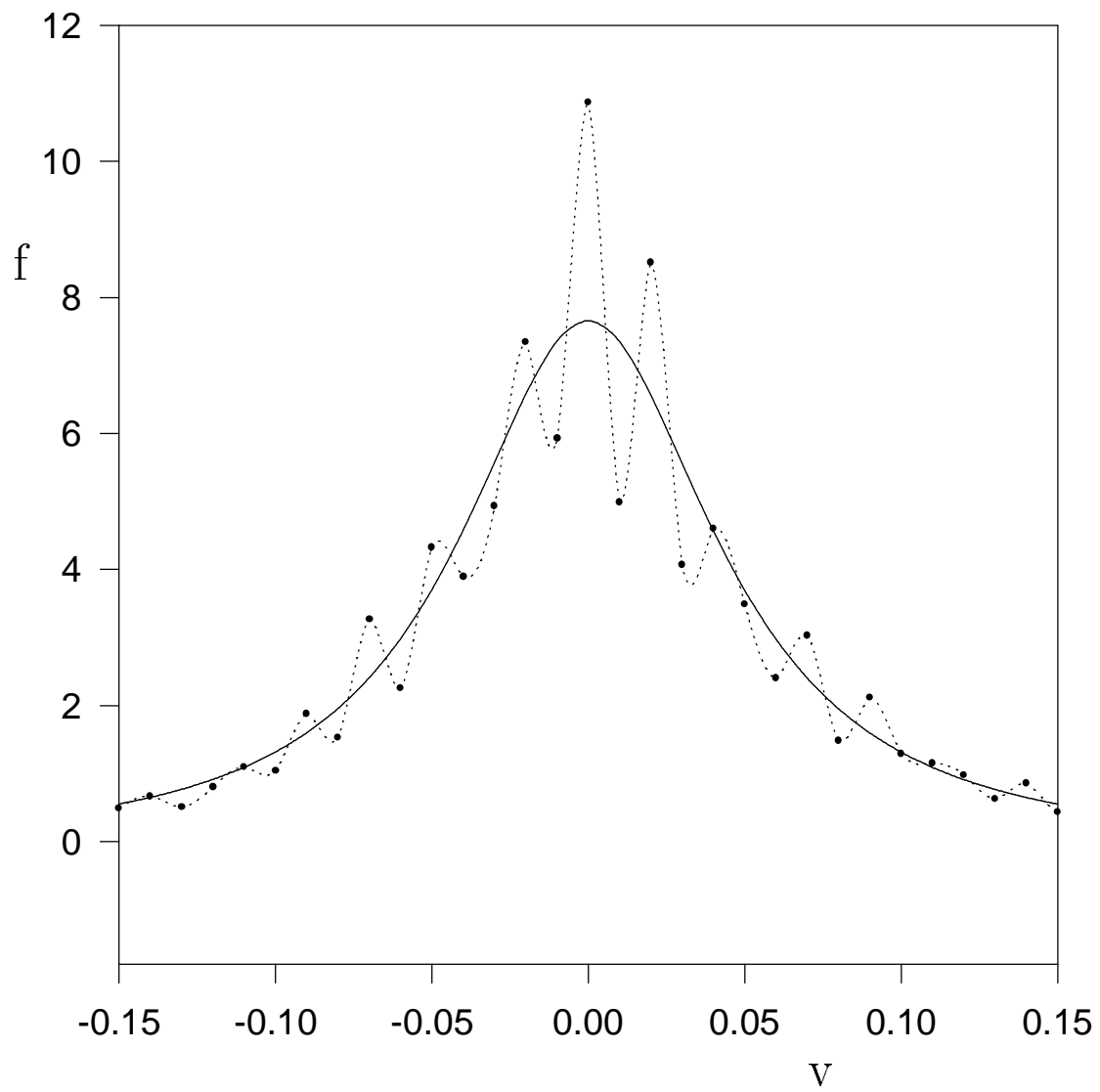


Figure 4a

Gaussian Density Compared to the Data

$$[m] = [1] , L = 1$$

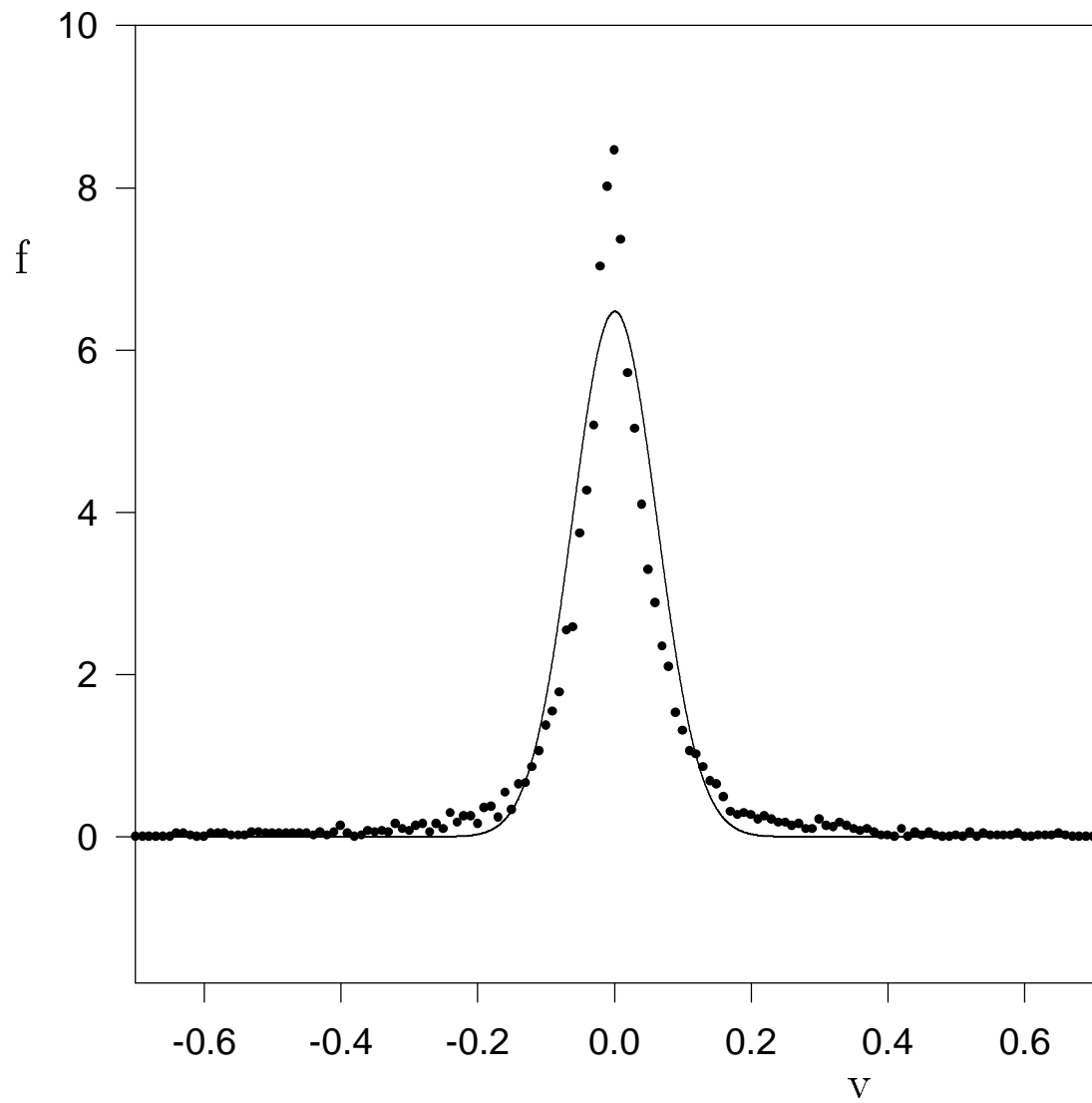


Figure 4b

Density Difference between
the Data and Gauss

$$[m] = [1], L = 1$$

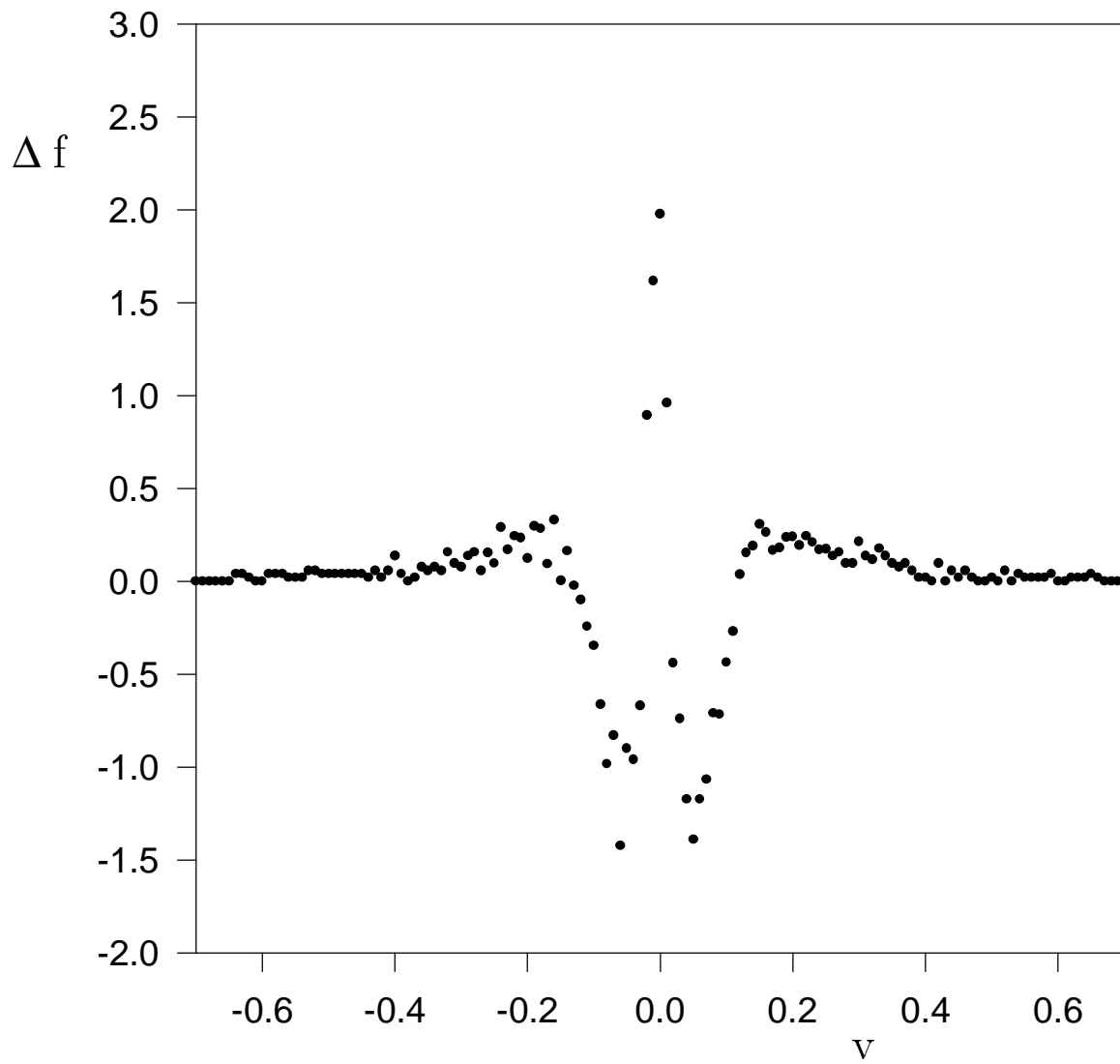


Figure 4c

Ratio R of the Data to the
Gaussian densities for $[m] = [1]$, $L = 1$

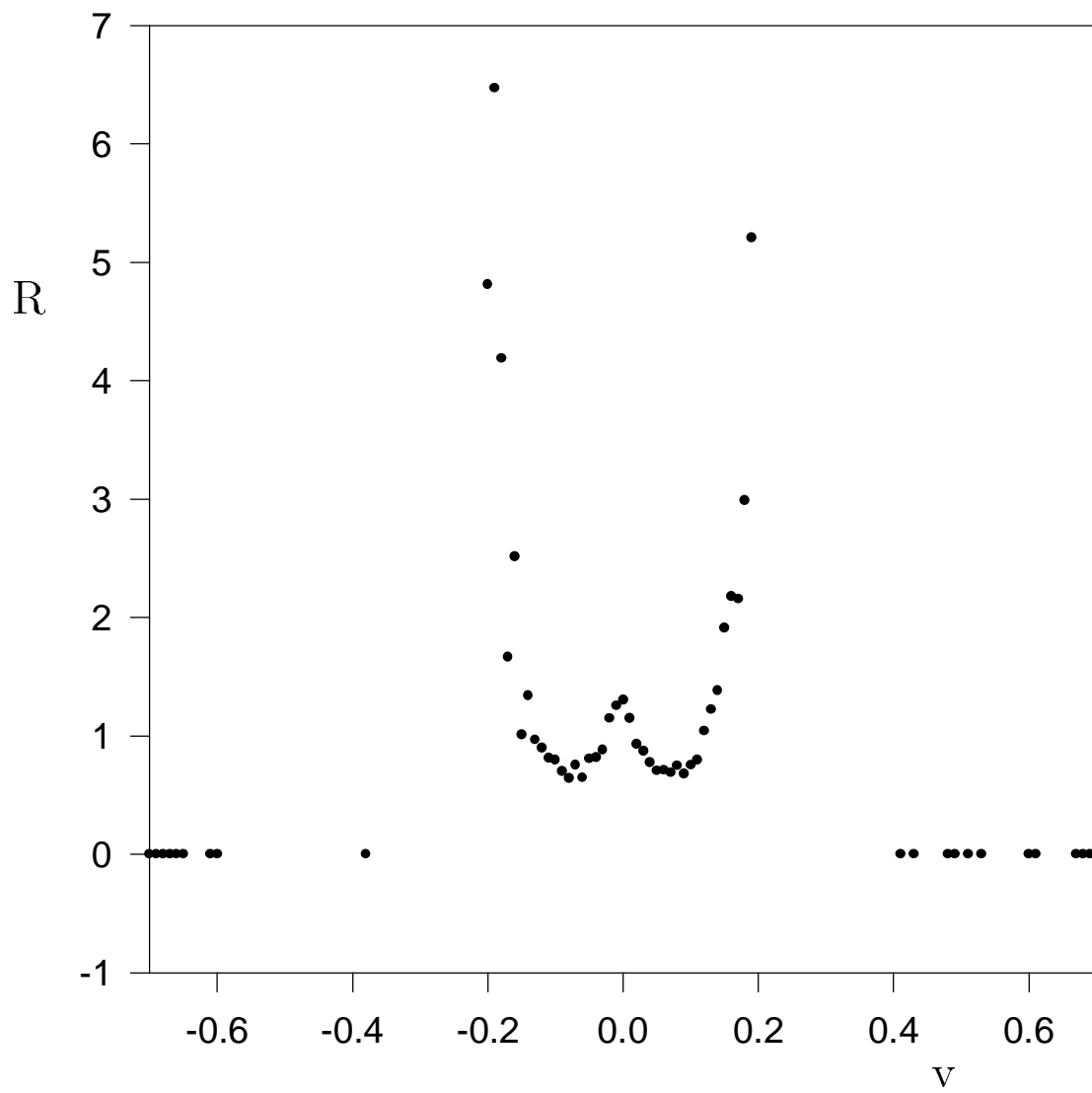


Figure 5

$\ln(q_1)$ versus $\ln(L/L_0)$

for $[m] = [1]$

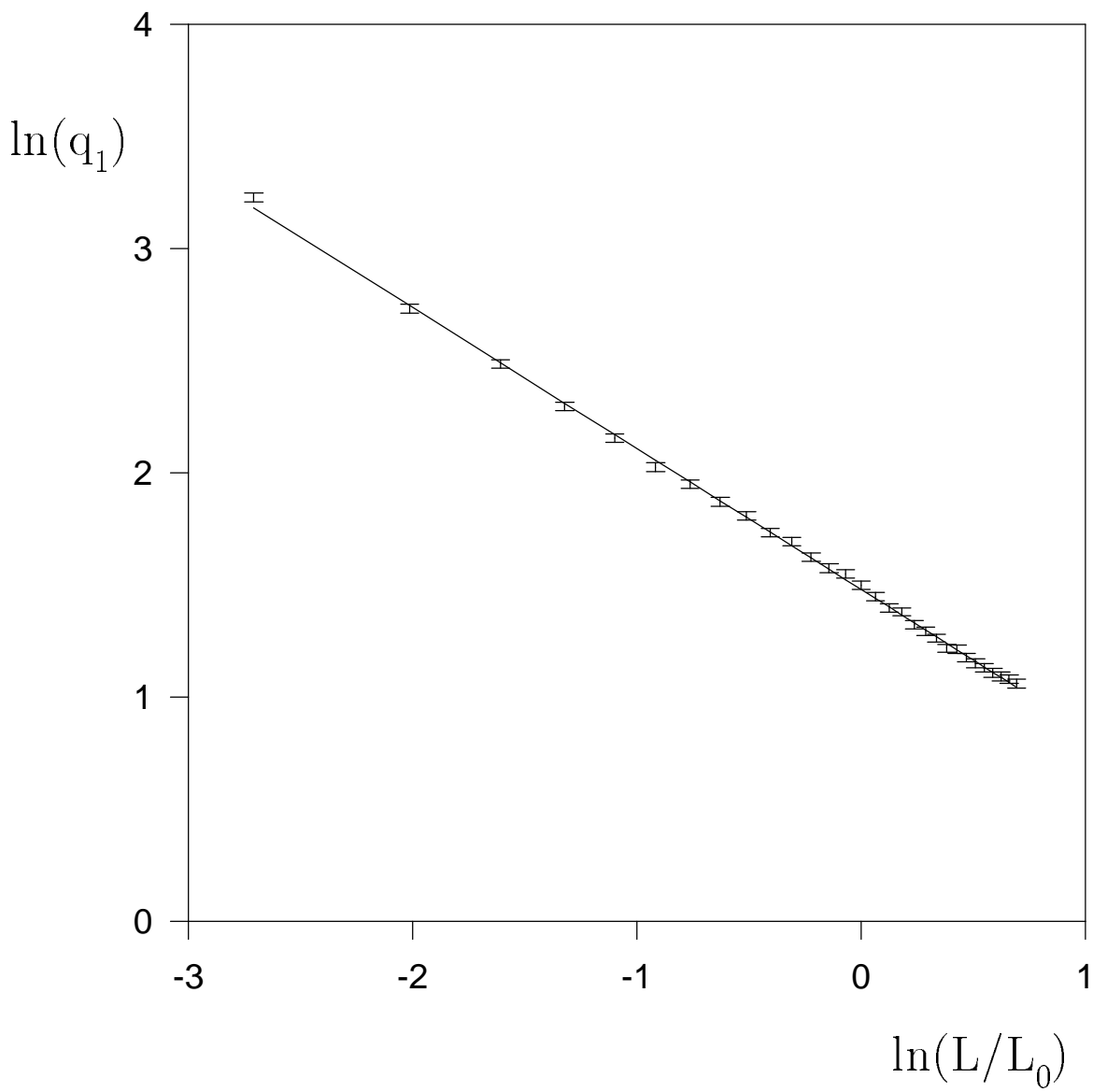


Figure 6

$\ln(-q_2)$ versus $\ln(L/L_0)$

for $[m] = [1]$

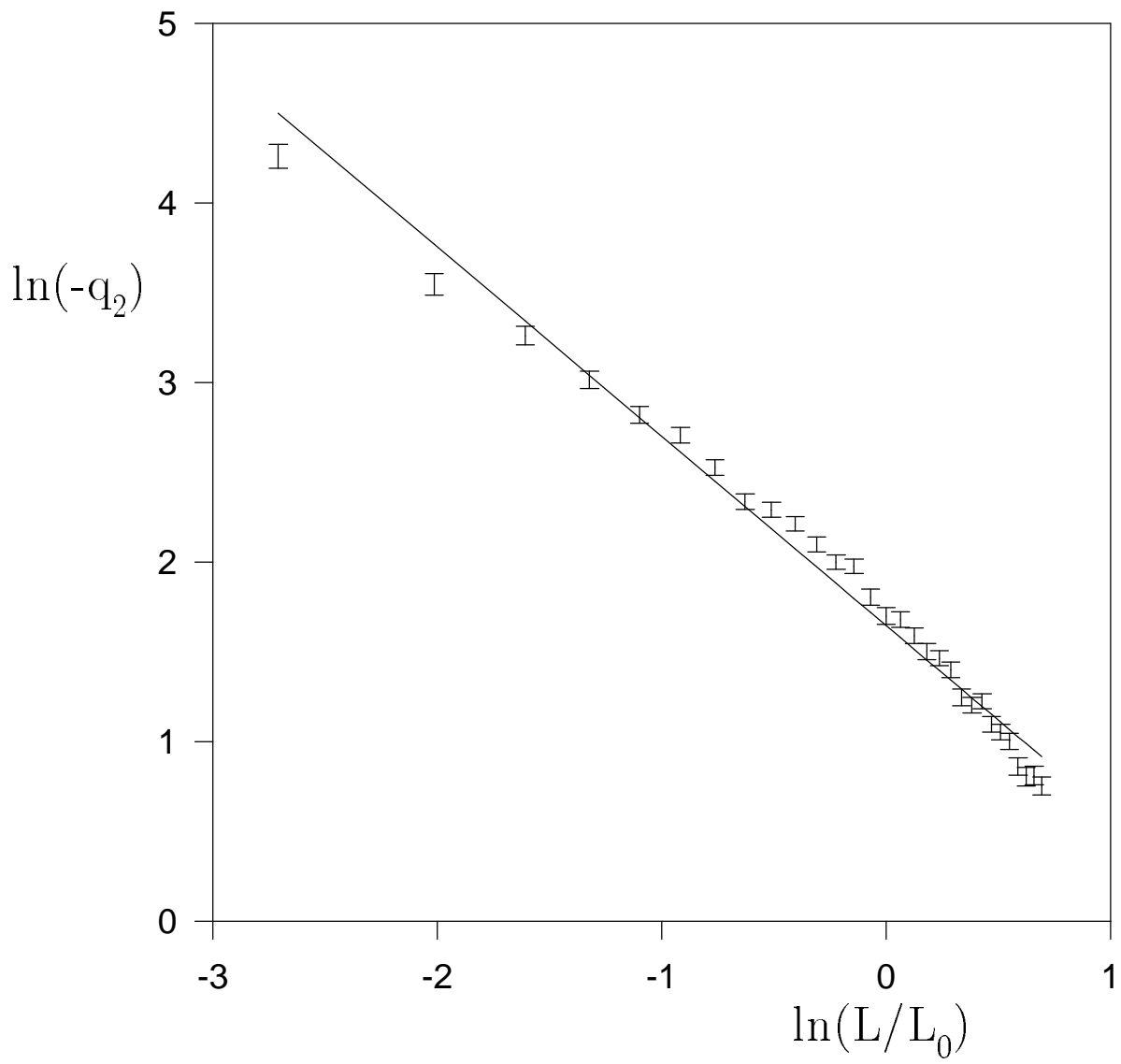


Figure 7

Confidence Region for q_1 and q_2

for $[m] = [1]$, $L = 1$

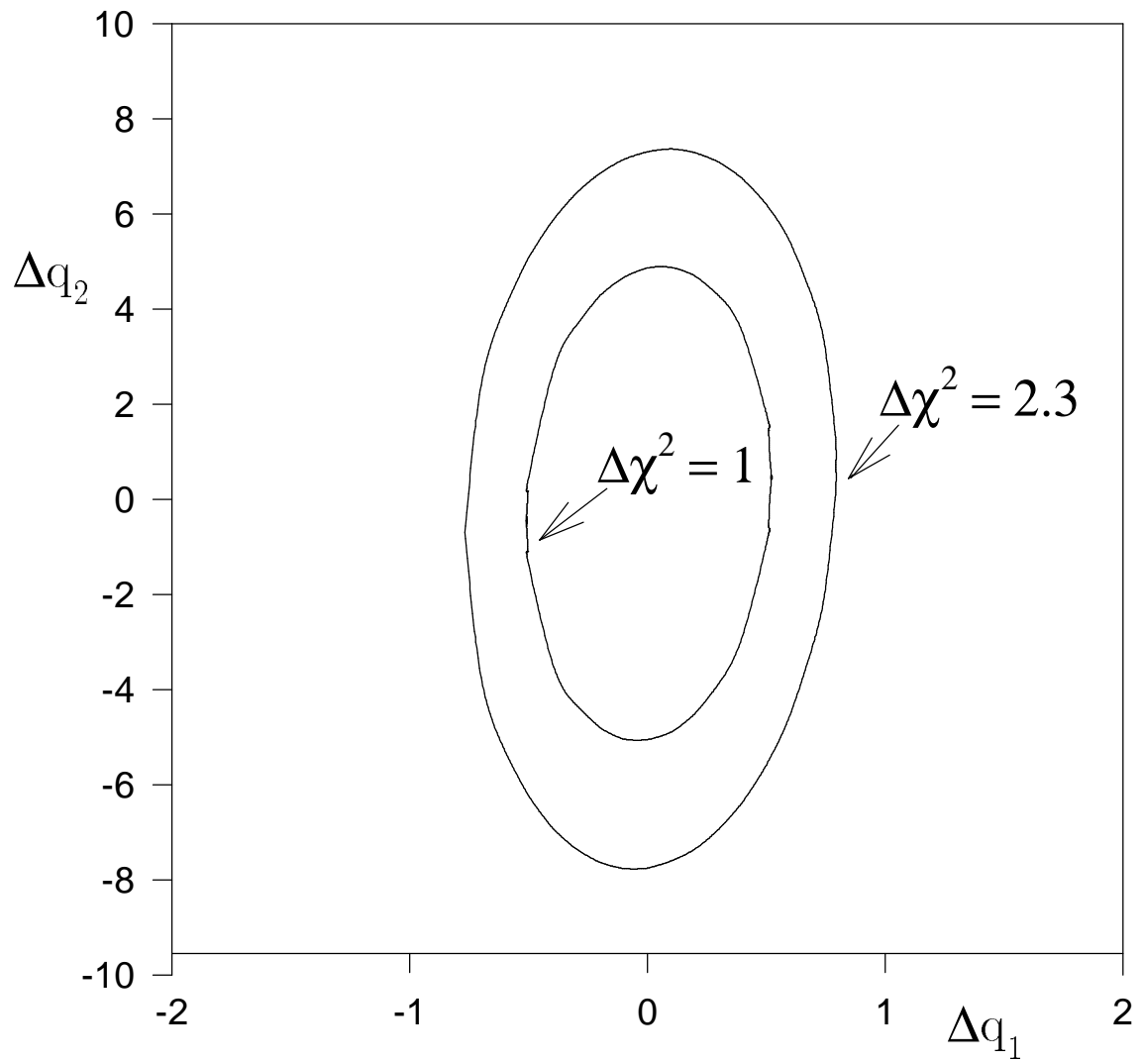


Figure 8

Confidence Region for λ_1 and \mathbf{v}_1

for $[\mathbf{m}] = [1]$

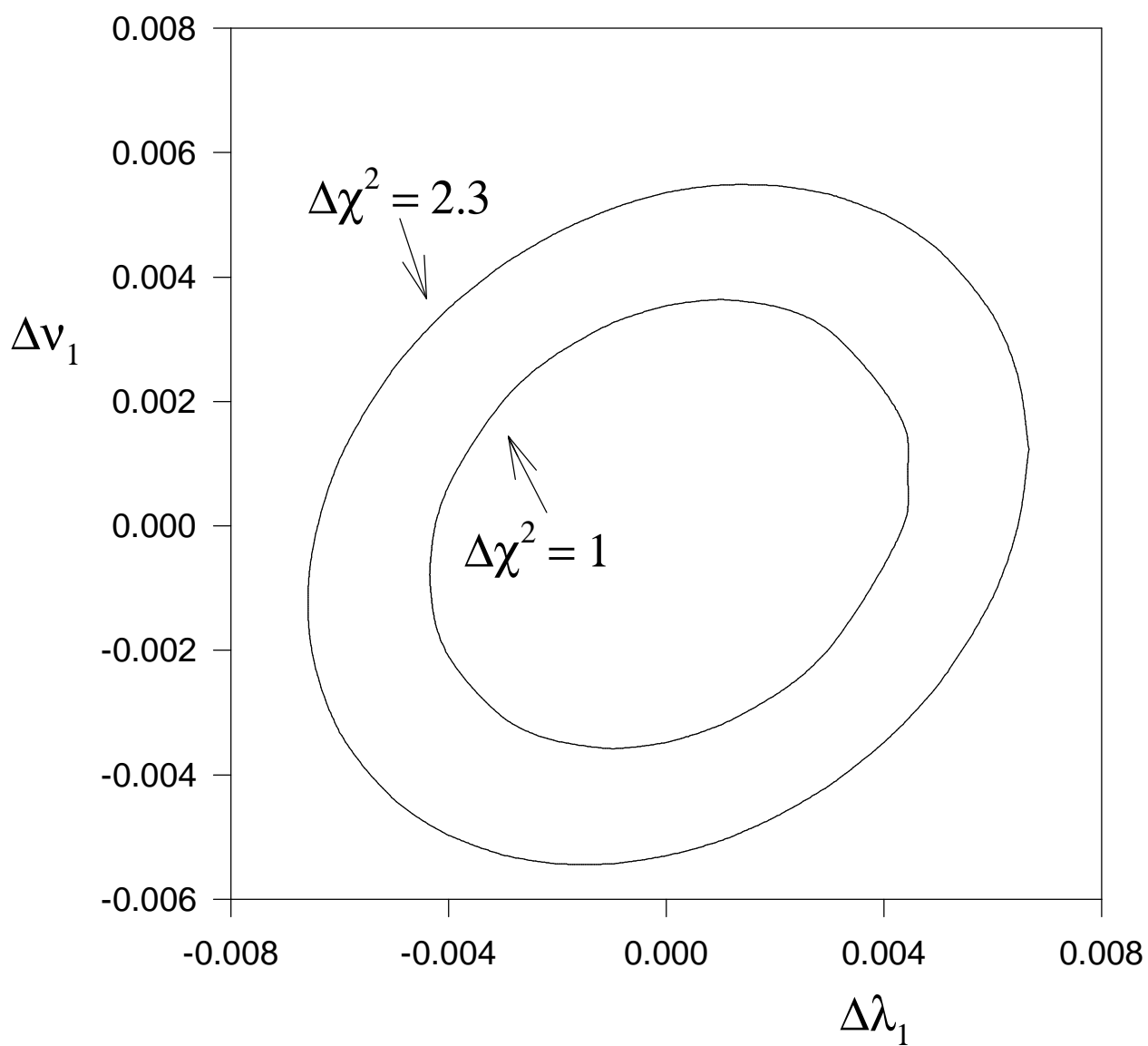


Figure 9

Confidence Region for λ_2 and \mathbf{v}_2

for $[\mathbf{m}] = [1]$

